

Cosmology

Prof. Dr. Matteo Maturi

Winter Semester 2025/26

About These Notes

- These lecture notes contain the material presented on the board.
- They are designed to guide you through the lectures content.
- While I primarily follow Bartelmann's script, these notes are complementary because:
 1. Some arguments are restructured, and I have selectively chosen the topics.
 2. Most computations are detailed step-by-step, leaving little to the imagination.
 3. Mathematical terms modified in the computations are marked with arrows for clarity.
 4. Instead of lengthy explanations, I use concise "slogans" to emphasize key points.
 5. The detailed discussion is provided during the lectures, where I am very "talkative".
- If you find any error, kindly let me know.

Notation, "Derivatives"

$\partial_x f \equiv \frac{\partial f}{\partial x} \equiv f_{,x}$	:	partial derivative with respect to x
df	:	total derivative
$\vec{\nabla} f$	:	gradient
$\vec{\nabla} \cdot \vec{f}$	:	divergence
$\vec{\nabla} \times \vec{f}$	:	curl
$\nabla^2 = \nabla^2$	:	Laplacian
$\vec{n} \cdot \vec{\nabla} f = \vec{\nabla}_{\vec{n}} f$	:	directional derivative
δS	:	functional derivative / Variation e.g. $\delta S = 0$
$\Delta f \equiv f(x+h) - f(x)$	:	difference

Differential geometry

$\nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma_{\lambda\mu}^\nu v^\lambda$	:	covariant derivative ∇ is clear from the context
$\mathcal{L}_X f$	:	Lie derivative
$d f$	:	exterior derivative

Figure 1: Some notation: take care in how I write variations and derivatives.

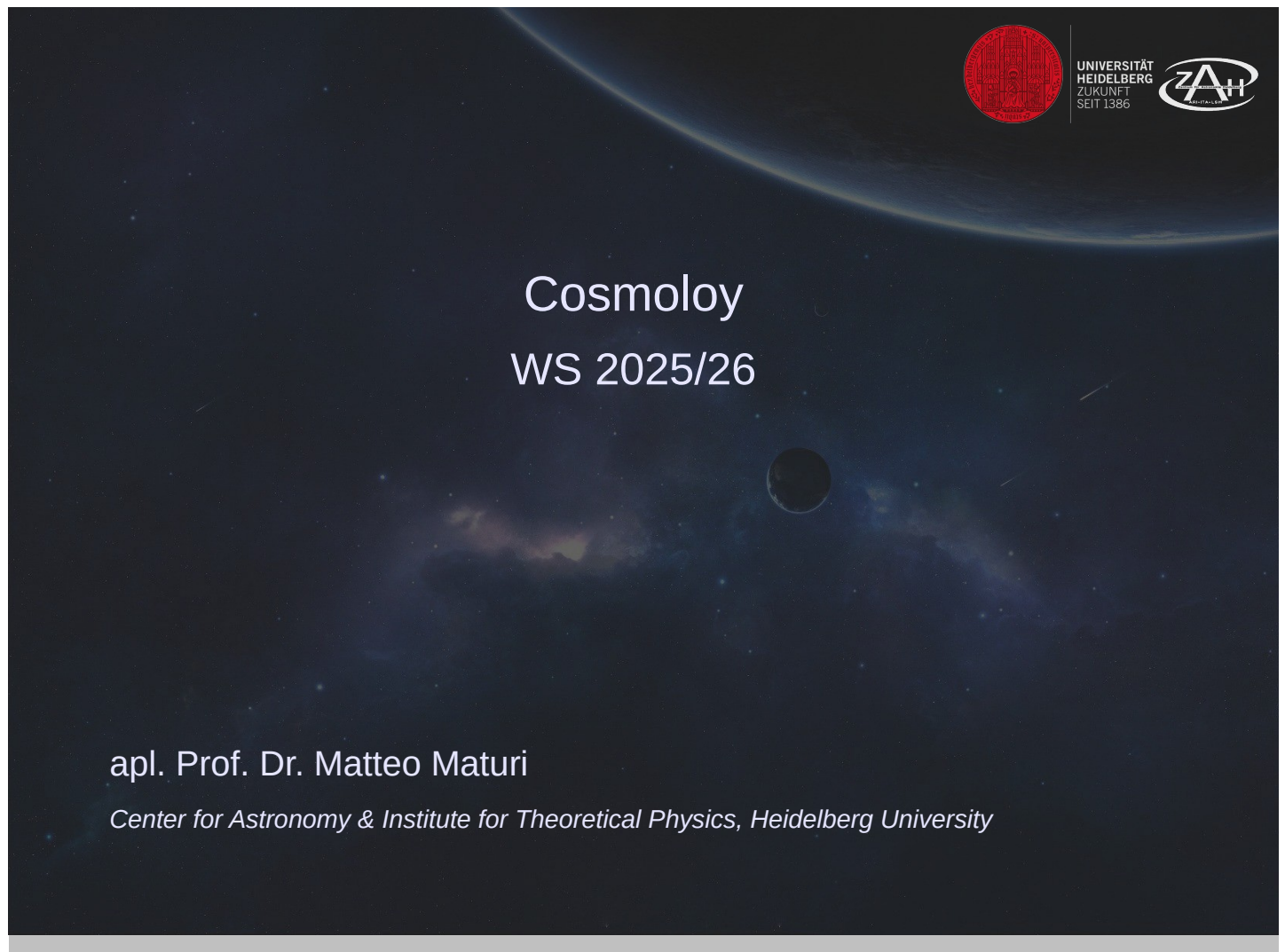
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Part I

Introduction



Nice to meet you!



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Lectures:

From October 16th to January 30th
Thursday 14:15-16:00 (INF308/HS 2)
Friday 14:15-16:00 (INF308/HS 2)

Website, Uebungen:

<https://uebungen.physik.uni-heidelberg.de/vorlesung/20252/2105>



- Enroll!
- Literature
- Lecture notes
- Additional material (slides, pdf files,...)
- Tutorials / Exercises
- All possible info

Lectures style

- In presence, mostly black board
- I will skip trivial steps...
...but, (all!) steps in lecture notes
- I will start super slow, I will accelerate
- Recap at beginning/end of each lectures
- Few mega recaps along the way
- Bonus: videos (posted with delay)

Material

- Lectures notes (what I present at the board)
- Scripts by Bartelmann, Amendola, Schaefer
- Books, e.g.
Cosmological Physics (John A. Peacock)
Modern Cosmology (S.Dodelson, F.Schmidt)

2.5 A collapsing Universe

Let's return to the basic equation (Eq. 2.6):

$$\ddot{R} = -\frac{GM}{R^2} \quad (2.18)$$

The first integration of this equation can be written in the form

$$\dot{R}^2 = \frac{2GM}{R} + C \quad (2.19)$$

where C is an integration constant. You can verify this by taking the d/dt of Eq. 2.19:

$$2\dot{R}\ddot{R} = -\frac{2GM}{R^2}\dot{R} \quad (2.20)$$

which gives Eq. 2.18 back. The integration constant C decides which of the three solutions you'll get. Eq. 2.19 in fact says that the kinetic energy equals the potential energy plus some constant.

If we seek a collapsing solution, there will be a time t_{ta} called the "turn around time" where the maximum $R = R_{\text{ta}}$ is reached and after which the solution collapses. The solution will be symmetric about this point in time. At this turn around time we will have $\dot{R} = 0$.

Let us rewrite Eq. 2.19 as

$$\dot{R} = \pm \sqrt{\frac{2GM}{R} + C} \quad (2.21)$$

Since, for the recollapsing solution, we know that $C < 0$ and that there will be a R_{ta} , for which

Cosmology, WS2023/24

01b Lorentz geome

Frame transformations

We deal with transformation between inertial frames

$$x^{\mu} \rightarrow x'^{\mu} \quad \begin{cases} * \left\{ \begin{array}{l} \frac{dx^{\mu}}{d\lambda} = \text{const.} \\ \frac{dx^{\mu}}{d\lambda} = \text{const.} \end{array} \right. & \text{(shift of coordinates through a point)} \\ ** \left\{ \begin{array}{l} \frac{d^2x^{\mu}}{d\lambda^2} = 0 \\ \frac{d^2x^{\mu}}{d\lambda^2} = 0 \end{array} \right. & \text{(no acceleration)} \end{cases} \quad \lambda: \text{affine parameter } x^{\mu}$$

Relation between frames S, S'

$$x'^{\mu} = X'^{\mu}(x^{\mu}(\lambda))$$

↑ transformations law
↑ transformed component

$$\frac{dx'^{\mu}}{d\lambda} = \frac{d}{d\lambda} \left(\frac{dx'^{\mu}}{d\lambda} \right) = \frac{d}{d\lambda} \left(\frac{\delta x'^{\mu}}{\delta x^{\alpha}} \frac{dx^{\alpha}}{d\lambda} \right) = \frac{\delta^2 x'^{\mu}}{\delta x^{\alpha} \delta x^{\beta}} \frac{dx^{\alpha}}{d\lambda} \frac{dx^{\beta}}{d\lambda} + \frac{\delta x'^{\mu}}{\delta x^{\alpha}} \frac{d^2 x^{\alpha}}{d\lambda^2} \stackrel{** \neq 0 \text{ (inertial frame)}}{=} 0 \Rightarrow \boxed{\frac{\delta^2 x'^{\mu}}{\delta x^{\alpha} \delta x^{\beta}} = 0}$$

* constant

Bureaucracy



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No uni-id?

- Write to Francesco to enroll you by hand

Tutorials:

- First tutorial next week!
- The exercises will not be corrected and no mark will be given
- It is possible to hand in exercises to get a feedback
- Discuss, discuss, discuss!!

Exam:

Written

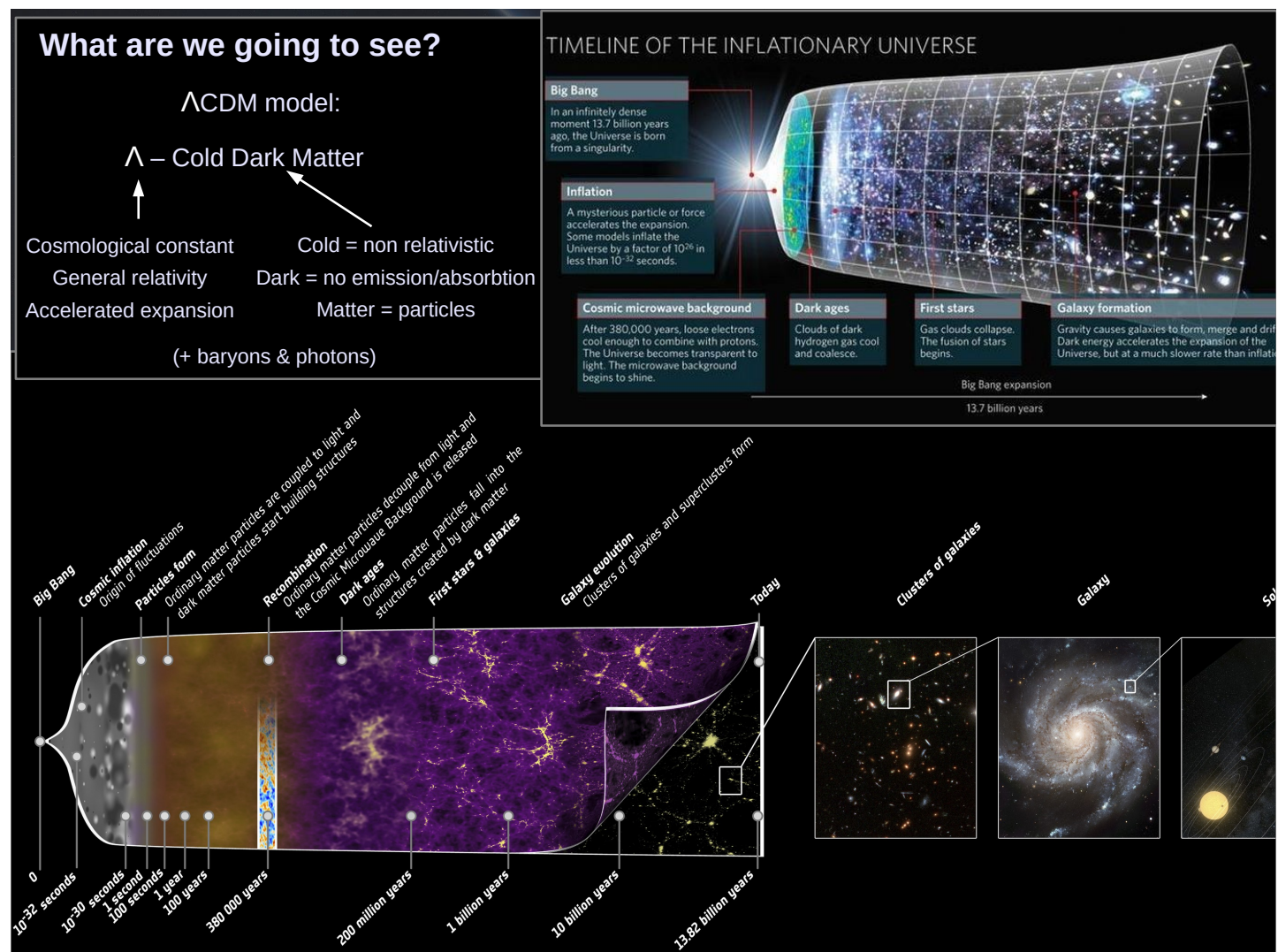
Same style of exercises

Admission to the exam: (50% attendance + (3 points)

- 1) attend at least 70% of the tutorials (your presence will be registered).
If attendance < 70%, it is required to hand in 3 full exercise sheets that will be graded.

AND

- 2) gain 5 points by:
 - 1 point: present at the black board at least 1/3 of a sheet.
 - 1 point: actively participating in the discussion during the tutorials

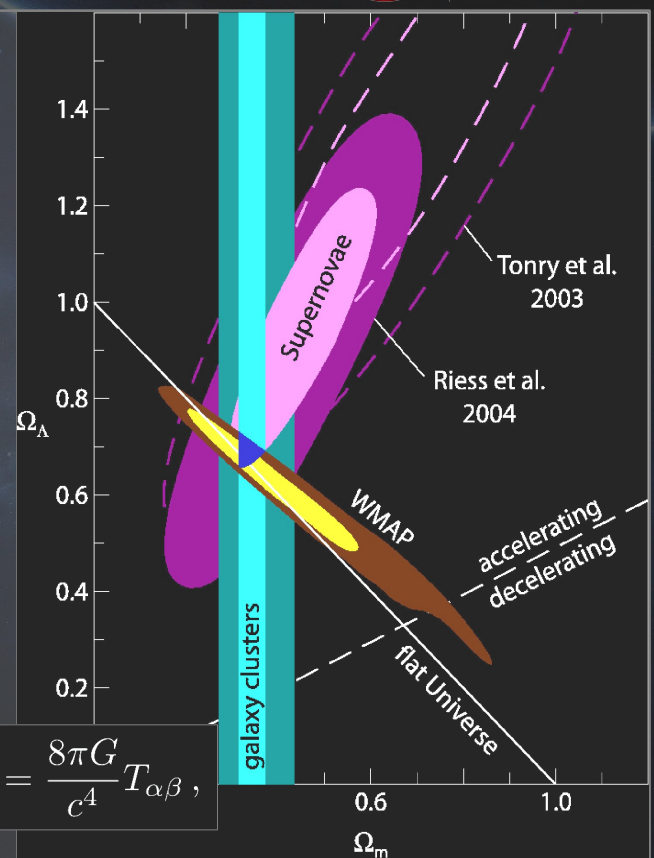


Why this cosmological model?

Data say so!

- Cosmic Microwave Background
- Supernovae luminosity distances
- Gravitational lensing
- Galaxy clusters
- Galaxy clustering
-

Dark energy + Dark matter?
or
Is gravity wrong?



$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = \frac{8\pi G}{c^4}T_{\alpha\beta},$$

How do we approach it?

Homogeneity and isotropy



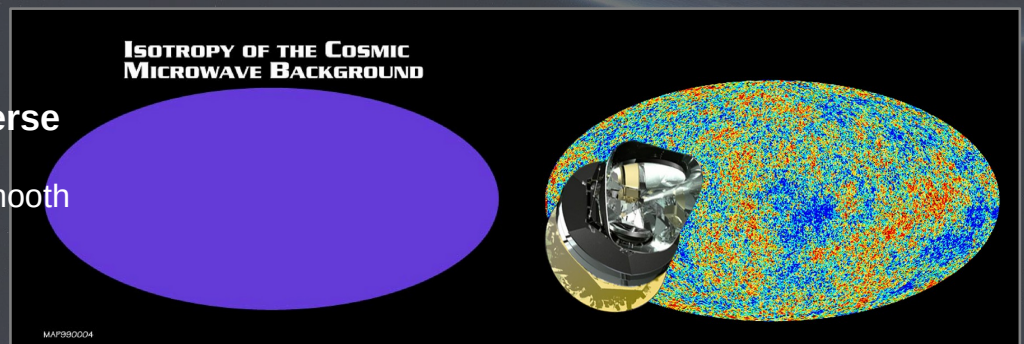
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The young universe

Very uniform and smooth

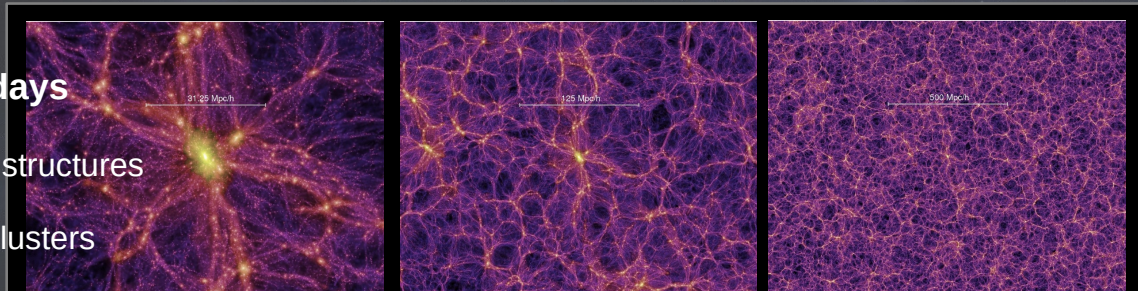
$$\Delta T/T = 10^{-5}$$



Now days

Large scale structures

Galaxy clusters



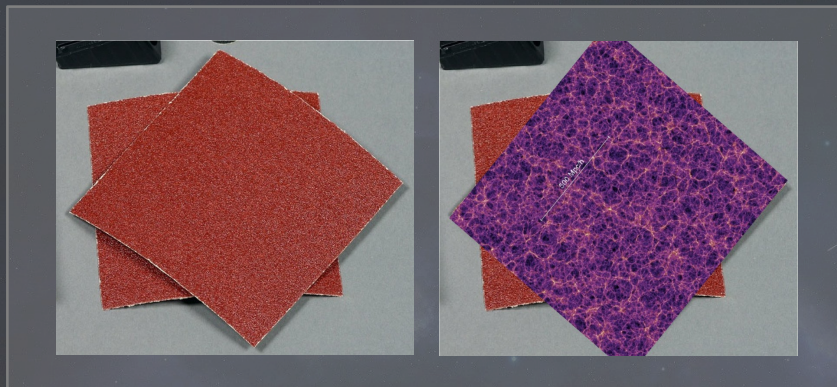
Let's build a model of the universe



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The universe is like sandpaper: rough but homogeneous and isotropic



To describe sand paper we do not need to describe each single grain!

3 parameters are enough:
material (M), grains dimension (D), grains number density (N)

The same for the universe...

The ingredients of the cake

Cosmological parameters

Dynamics:

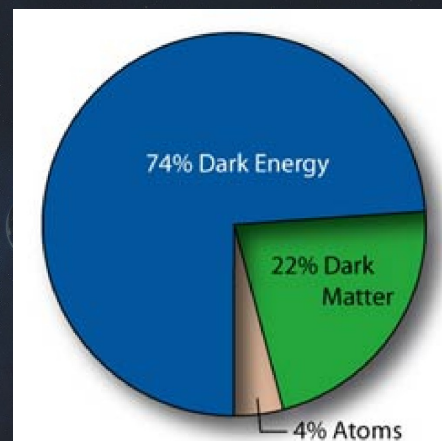
- H_0 : Hubble parameter

Ingredients:

- Ω_r : radiation density
- Ω_b : barionic density
- Ω_d : dark matter density
- Ω_Λ : dark energy density

"Raffness":

- σ_8



And the roles of the game?

How do the ingredients interact?



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The rules mastering the universe

Theory	Interaction	Behavior	Typical scale
QCD	Strong	$1/r^7$	$1.4 \times 10^{-15} \text{ m}$
Electroweak	Weak	$1/r^5$ to $1/r^7$	10^{-18} m
QED	Electromagnetic	$1/r^2$	∞
GR	Gravity	$1/r^2$	∞

We need a long distance force acting on a neutral universe
Gravity!!

Newtonian cosmology



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Consider a spherical portion of the Universe

- radius $l = ar_0$
- density ρ
- mass $M = \rho \frac{4}{3} \pi l^3$
- kinetic energy $A = \frac{kc^2}{2}$ init. condit.



$$\text{Acc.} = \ddot{a} = -\frac{GM}{l^2}$$

With the previous equations we obtain:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho) \quad \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$$

Relativistic cosmology



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Einstein's field equation

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = \frac{8\pi G}{c^4}T_{\alpha\beta},$$

$$ds^2 = (cdt)^2 - a^2(t) \left[\frac{dr^2}{(1 - kr^2)} + r^2(d\theta^2 + \sin^2\theta d\varphi) \right]$$

Friedmann–Lemaître–
Robertson–Walker metric

$$T_{\alpha\beta} = -pg_{\alpha\beta} + (p + \rho c^2) \mu_{\alpha}\mu_{\beta} - \frac{c^4\Lambda}{8\pi G}g_{\alpha\beta}.$$

Universe ingredients

Friedmann's equations

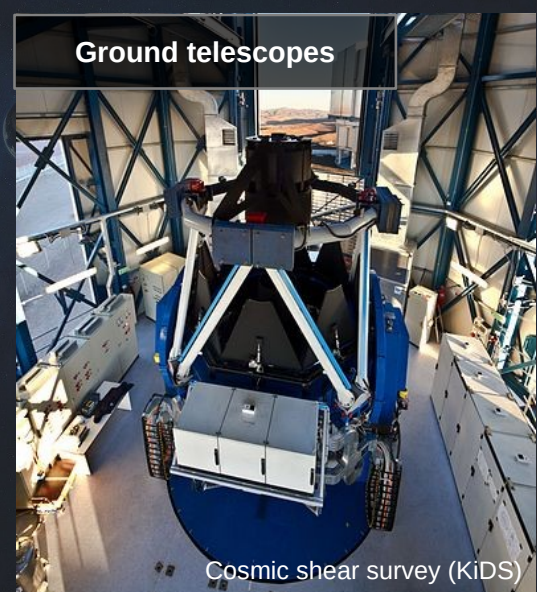
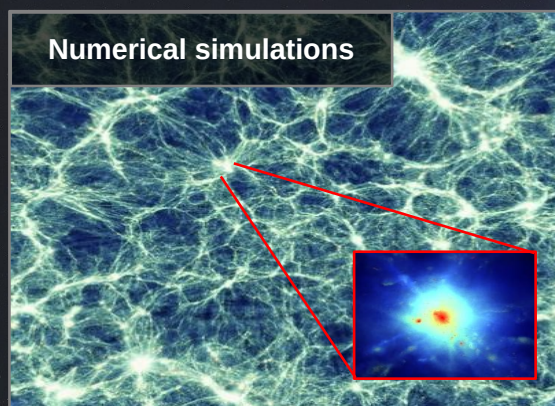
$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3} \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda}{3} \end{aligned}$$

Newtonian

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho) \end{aligned}$$

The tools to explore the universe

Analytical: $8+1 \sim 10$





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Detailed content of the lectures

1) Introduction

- Special Relativity
- General Relativity

2) Uniform universe

- FLRW metric
- Friedmann equations
- Scalar fields
- Dark energy
- Cosmological inflation

3) Thermal History

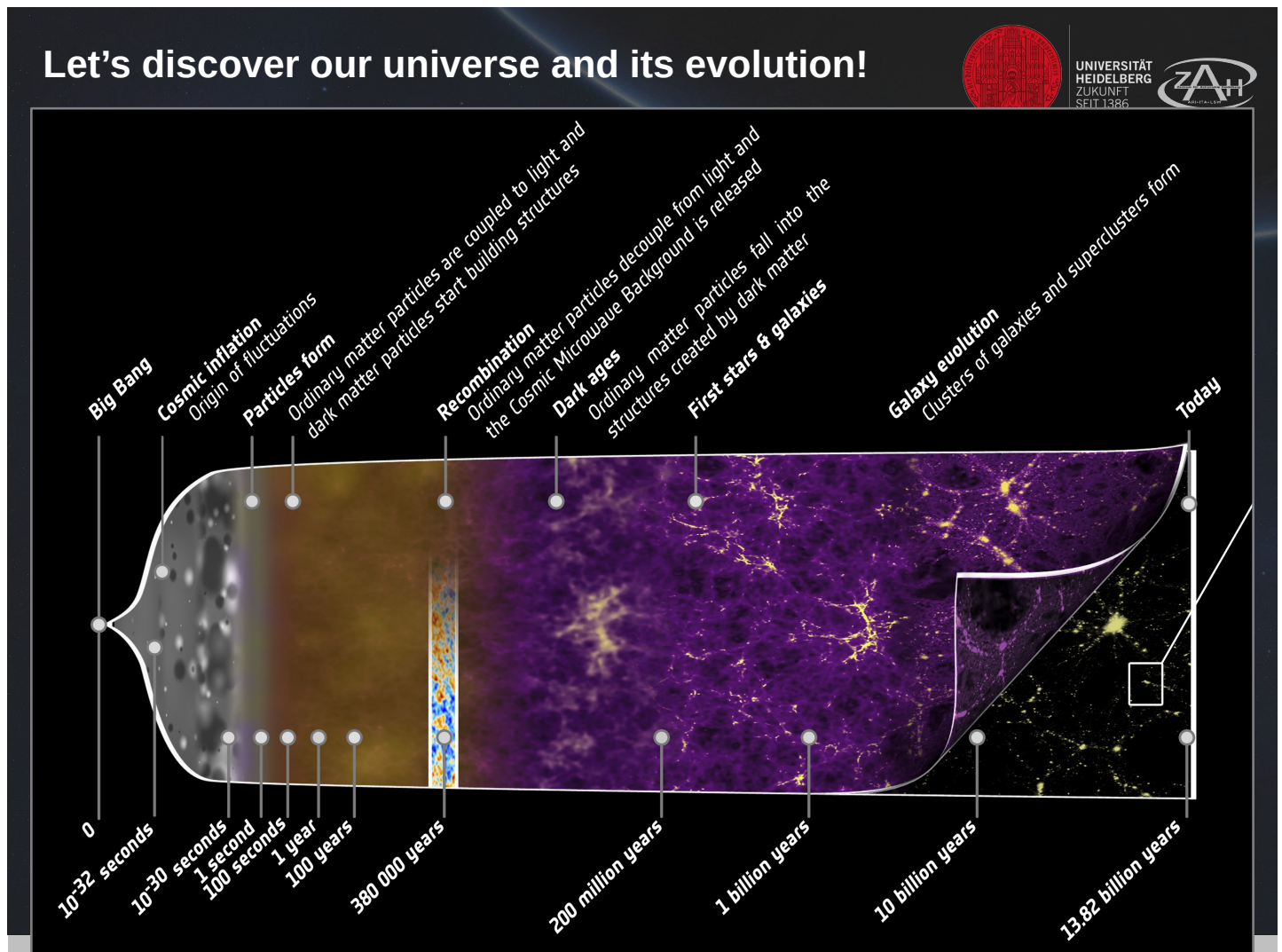
- Summary of equilibrium thermodynamics
- When reactions starts/end
- Neutrinos decoupling (CvB)
- Cosmological Nucleosynthesis
- Electrons decoupling (CMB)
- Cosmic reionization

4) Structure formation

- Hydrodynamics (Continuity, Euler, Poisson)
- Linear theory
- Dark matter halos
- N-Body numerical simulations

5) Current Observations

- Galaxy clustering
- Cosmic Microwave Background (CMB)
- Gravitational lensing
- Galaxy clusters
- Other probes...



Special Relativity: the core!

- $P = (t, x, y, z)$ Event: something happening in P
- S, S' Coordinate frames (in standard configuration)
- Generic linear transformation (preserve homogeneity!)

$$x' = At + Bx, \quad y' = y, \quad z' = z$$

$$t' = Dt + Ex$$

Find coefficients:

$$\begin{cases} x=0 \Rightarrow x' = At & \frac{x'}{t'} \equiv v' = \frac{A}{D} & \boxed{A = Dv'} \\ x'=0 \Rightarrow 0 = At + Bx = At + Bvt & A = -Bv \end{cases}$$

$\xrightarrow{-v} Dv' = -Bv \Rightarrow \boxed{D=B}$

$$\boxed{x'} = \overbrace{Dv'}^A t + \overbrace{D}^B x = D(v't + x) = \boxed{D(x - vt)}, \quad y' = y, \quad z' = z$$

$$\boxed{t' = Dt + Ex}$$

- D, E are undefined $\Rightarrow$ impose an additional condition: 2 possibilities...

a) Postulate: $t' \stackrel{!}{=} t$ (Absolute time, Galilean frame)

$$t' = Dt + Ex \quad \forall t, x \Leftrightarrow D=1 \quad E=0 \quad \Rightarrow \quad \boxed{\begin{matrix} x' = x - vt \\ t' = t \end{matrix}}$$

b) Postulate: $c \stackrel{!}{=} \text{const} \quad \forall \text{ frame}$ (observations)

- $c = \text{const} \Rightarrow$ it makes sense to investigate light

- Consider a photon in 2 frames

$$c^2 t^2 = x^2 + y^2 + z^2 \quad \Rightarrow \quad 0 = -c^2 t^2 + x^2 + y^2 + z^2$$

$$c^2 t'^2 = x'^2 + y'^2 + z'^2 \quad \Rightarrow \quad 0 = -c^2 t'^2 + x'^2 + y'^2 + z'^2$$

$$\Rightarrow \quad \boxed{-c^2 t^2 + x^2 + y^2 + z^2 = -c^2 t'^2 + x'^2 + y'^2 + z'^2}$$

in a cartesian frame!

Plug $\otimes$

$$\Rightarrow \quad \boxed{\begin{matrix} ct' = \gamma(ct - \beta x) & y' = y \\ x' = \gamma(x - \beta ct) & z' = z \end{matrix}} \quad \gamma = (1 - \beta^2)^{-1/2} \quad \beta = \frac{v}{c} \quad \text{Lorentz transformations}$$

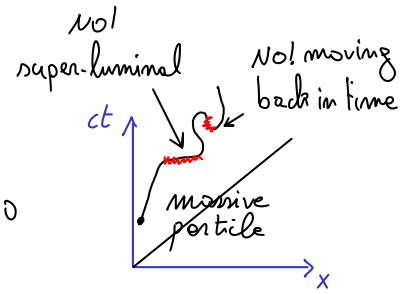
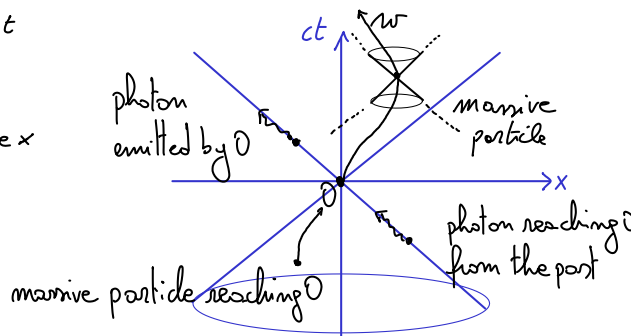
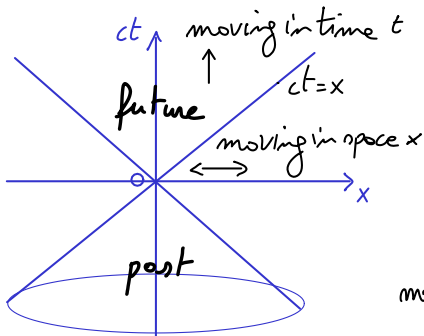
$\hookrightarrow$ Space and time are connected "Not like the frames of a movie"

$\hookrightarrow$ The Space-Time! $\Rightarrow$ event $P \equiv (x^\mu) \equiv (ct, x, y, z)$ ct to have units of length
 x^μ elements of a 4-vector

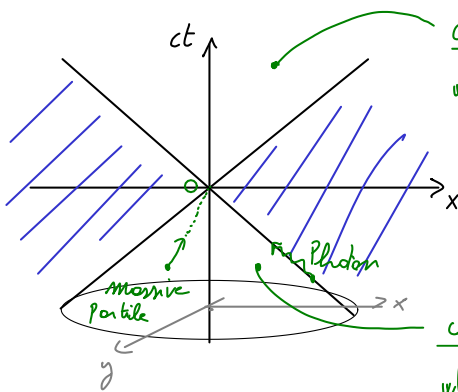
4-interval: $d\bar{s}^2 = -(\underbrace{cdt}_{dx^0})^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 = \sum_i \sum_j \eta_{ij} dx^i dx^j \quad \eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ Minkowski metric (!)

Space-time diagrams

Space-time trajectories (world-lines)



Causal connection (Horizon)

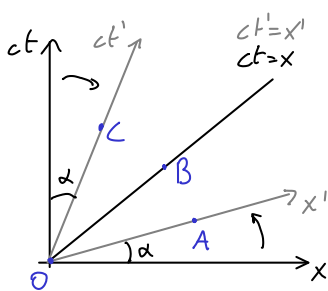


causality:
what O can get in contact to in the future

Causal connection
within the light cone

causality:
what can get in contact with O from the past

Types of events



Lorentz transf. are "rotations" (boosts) $\frac{dx}{dt} = \frac{v}{c} \Rightarrow \alpha = \arctan\left(\frac{v}{c}\right)$

OA: Space like event $ds > 0$ (frame in which $\Delta t' = 0$)

OB: null $ds = 0$

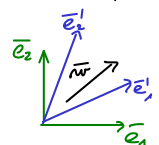
OC: Time like event $ds < 0$ (frame in which $\Delta x' = 0$)

Note: ds^2 is NOT positive definite

Because of postulate: • null line do not change under Lorentz. transf.

• ds^2 is invariant under Lorentz. transf. (i.e. is the same $\forall$ frame)

"The norm of a vector does not depend on the basis"



Some ambiguity is left: "signature" choice

$$\left(\text{e.g. } -c^2 dt^2 + d\vec{r}^2 = 0 \quad \text{or} \quad c^2 dt^2 - d\vec{r}^2 = 0 \right)$$

For photons $\rightarrow \phi(v) \cdot ds^2 = 0, ds^2 = 0$
 $\rightarrow ds^2 = 0, \phi(v) \cdot ds^2 = 0$ with $\phi \in \mathbb{R}$

$$\Rightarrow ds^2 = \phi(v) ds^2$$

$$\Rightarrow ds^2 = \phi(v) ds^2 = \phi^2(v) ds^2 \Rightarrow \boxed{\phi(v) = \pm 1}$$

Here we choose $(-, +, +, +)$ signature

2 possible "signatures"

The proper time

• Time is measured with a clock at rest in a frame

$$\underbrace{d\vec{r}^2 \equiv 0}_{\text{(define frame center)}} \quad -c^2 dt'^2 + dx'^2 + dy'^2 + dz'^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

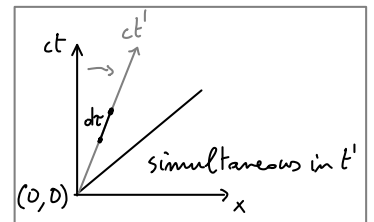
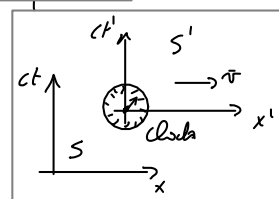
$$\updownarrow \quad \underline{dt' \equiv d\tau \text{ proper time}}$$

$$-c^2 d\tau^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 = ds^2 \Rightarrow \boxed{d\tau^2 = -\frac{ds^2}{c^2}} \quad (1)$$

$$= -c^2 dt^2 \left(1 - \frac{dx^2 + dy^2 + dz^2}{c^2 dt^2} \right) \Rightarrow \boxed{d\tau = \gamma^{-1} dt} \quad (2) \quad dt = \gamma d\tau$$

time dilation

$$\underbrace{\left(1 - \frac{\vec{v}^2}{c^2} \right)}_{(1-\beta^2)} = \gamma^{-2}$$



• Time interval : $\Delta \tau = \int_{\omega} d\tau = \int_{\omega} \gamma^{-1} dt$

$$\stackrel{(1)}{=} \int_{\omega} \frac{1}{c} \sqrt{-ds^2} = \int_{\omega} \frac{1}{c} \sqrt{-\eta_{\mu\nu} dx^{\mu} dx^{\nu}}$$

The metric is "hidden" there!

(!) in G.R $\eta_{\mu\nu} \rightarrow g_{\mu\nu}(x^{\alpha})$ local properties of space-time

Implication of the Lorentz transforms

$$\cdot ct' = \gamma(ct - \frac{v}{c}x) \quad x' = \gamma(x - vt) \quad y' = y \quad z' = z \quad (\text{Standard configuration of } S, S')$$

• Time dilation

$$t_0 \equiv t_B - t_A \quad (\text{proper time})$$

$$t = t_0(1 - \beta^2)^{-1/2} = t_0 \gamma \quad \Rightarrow \quad v \uparrow \Rightarrow t \uparrow$$

• Length contraction

$$l_0 \equiv x'_0 - x'_1 \quad (\text{proper length})$$

$$l = l_0(1 - \beta^2)^{1/2} = l_0 \gamma^{-1} \quad (v \uparrow \Rightarrow l \downarrow)$$

$$\text{what if } v = c? \quad \Rightarrow \quad l \rightarrow 0$$

$$\Rightarrow \quad \gamma = \frac{m}{l^3} = \infty!$$

($\Rightarrow v = c$ not possible for massive particles)

• Velocity transformation

$$u'_x \equiv \frac{dx'}{dt'} = \frac{dx - v dt}{dt - \frac{v}{c^2} dx} = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}$$

(in lab frame: $u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}}$)

$$u'_y \equiv \frac{dy'}{dt'} = \frac{u_y}{\gamma(1 - \frac{u_x v}{c^2})} \quad (!)$$

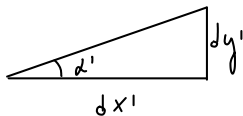
$$u'_z \equiv \frac{dz'}{dt'} = \frac{u_z}{\gamma(1 - \frac{u_x v}{c^2})} \quad (!)$$

because the time is affected

• Accelerations

$$a'_x \equiv \frac{du'_x}{dt'} \quad \dots$$

• Angles

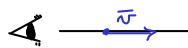


$$dy' = dx' \tan \alpha' \Rightarrow$$

$$\tan \alpha' = \frac{dy'}{dx'} = \frac{dy}{\gamma(dx - v dt)} = \frac{dy}{\gamma dt (\frac{dx}{dt} - v)} = \frac{u_y}{\gamma(u_x - v)}$$

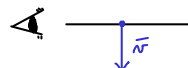
• Doppler effect

↳ longitudinal:



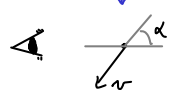
$$\frac{\nu}{\nu'} = \left(\frac{1 - \beta}{1 + \beta} \right)^{1/2}$$

↳ transverse:



$$\nu = \nu' \gamma^{-1} \quad (t = t') \quad \text{purely relativistic effect}$$

↳ Arbitrary direction:



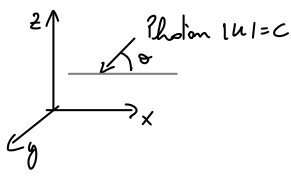
$$\frac{\nu}{\nu'} = [\gamma(1 + \beta \cos \theta)]^{-1}$$

↑
angle seen by the observer

$$\frac{\nu}{\nu'} = \gamma(1 - \beta \cos \theta')$$

↑
angle in rest frame of moving source

• Aberration of light



Polar coordinates:

$$\begin{aligned} u'_x &= c \cos \theta' & u'_y &= 0 & u'_z &= c \sin \theta' \\ u_x &= c \cos \theta & u_y &= 0 & u_z &= c \sin \theta \end{aligned}$$

Lorentz transf. of velocities $\rightarrow$

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}$$

$$= \frac{c \cos \theta - v}{1 - \beta \cos \theta} \Rightarrow$$

$$\cos \theta' = \frac{\cos \theta - \beta}{1 - \beta \cos \theta}$$

in lab frame: $\cos \theta = \frac{\cos \theta' + \beta}{1 + \beta \cos \theta'}$

• Beaming of light : (solid angles)

$$d\Omega' = d\phi' d\cos \theta'$$

$$d\Omega = d\phi d\cos \theta : d\phi = d\phi' \quad \text{no change because } \perp \text{ to velocity}$$

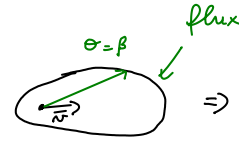
$$d\cos \theta = d\left(\frac{\cos \theta' + \beta}{1 + \beta \cos \theta'}\right) = \frac{d\cos \theta'}{\gamma^2 (1 + \beta \cos \theta')^2}$$

$$= \frac{d\phi d\cos \theta'}{\gamma^2 (1 + \beta \cos \theta')^2}$$

super relevant for emission processes:



isotropic emission in source rest frame



$\Rightarrow$ privileged direction of emission

anisotropic emission for observer

eg. $\theta' = \frac{\pi}{2}$: $\cos \theta' = 0 \Rightarrow \cos \theta = \frac{\cos \theta' + \beta}{1 + \beta \cos \theta'} = \beta$

$$\beta \ll 1 \Rightarrow \cos \theta \sim 0 \quad \theta \sim \pi/2$$

$$\beta \sim 1 \Rightarrow \cos \theta \sim 1 \quad \theta \sim 0 \quad (\text{beamed forward})$$

• Cross section

Solid angles are affected $\Rightarrow$ cross sections transform as well

Lorentz geometry

Geometric interpretation of the ct,x plane

- Define a vector space
- $\bar{x} = (x^\mu) = (x^0, x^1, x^2, x^3) = x^\mu \bar{e}_\mu \in \mathbb{M}$ 4-vector $x^0 = ct$ $\bar{e}_\mu =$ basis vector $\mathbb{M}$ Minkowski space
- Separations in the space-time are expressed by the 4-interval $d\bar{s}$
 $\Rightarrow$ Distances are measured in terms of a metric $\Rightarrow$ metric vector space

$$\begin{aligned}
 d\bar{s}^2 &= -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \quad \text{4-interval} \Rightarrow \text{vector space } d\bar{s} \in \mathbb{M} \text{ vector in Minkowski space} \\
 &= \sum_{\nu=0}^3 \sum_{\mu=0}^3 \eta_{\mu\nu} dx^\mu dx^\nu \quad \text{Minkowski metric} \quad \eta = \text{diag}(-1, 1, 1, 1) \quad \eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
 &= \eta_{\mu\nu} dx^\mu dx^\nu \quad \leftarrow \text{Einstein notation and index lowering} \\
 &= dx_\nu dx^\nu \\
 &= \bar{d}\bar{x}^T \eta \bar{d}\bar{x} \quad \text{matrix notation} \quad \text{Based on a frame} \Rightarrow \text{components} \\
 &= \eta(\bar{d}\bar{x}, \bar{d}\bar{x}) \quad \eta: \mathbb{M} \times \mathbb{M} \rightarrow \mathbb{R} \quad \eta(\bar{u}, \bar{u}) = a \in \mathbb{R} \quad \forall \bar{u}, \bar{v} \in \mathbb{M} \quad \text{Rank}(0) \text{ tensor} \quad \text{bilinear map identifying the scalar product} \\
 &= \langle \bar{d}\bar{x}, \bar{d}\bar{x} \rangle \quad \text{scalar product} \\
 &= \tilde{d}\bar{x}(\bar{d}\bar{x}) \quad \eta: \mathbb{M} \rightarrow \mathbb{M}^* \quad \eta(\bar{v}, -) = \tilde{v} = \bar{v}^* = (v_\nu) \in \mathbb{M}^* \quad \text{Dual space of } \mathbb{M}, \text{ linear map (isomorphism)} \\
 &\quad \tilde{v}: \mathbb{M} \rightarrow \mathbb{R} \quad \tilde{v}(\bar{u}) = \tilde{v}(\bar{u}) = v_\nu u^\nu = a \in \mathbb{R} \quad \text{dual vector, tangent vector, 1-form ...} \\
 &\quad \text{linear map mapping vectors in covectors} \\
 &= \|d\bar{s}\|^2 \in \mathbb{R} \quad \text{norm of } d\bar{s} \Rightarrow \text{invariant, it can be negative!}
 \end{aligned}$$

Useful expressions

η is symmetric

$$\eta_{\mu\nu} = \eta_{\nu\mu}$$

Inverse of the metric

$$\eta^{\mu\alpha} \eta_{\nu\alpha} = \eta^\mu{}_\nu = \delta^\mu_\nu \quad \delta^\mu_\nu \text{ Kronecker delta (identity)}$$

Lowering, raising indices

$$\eta: \mathbb{M} \rightarrow \mathbb{M}^* \quad v_\mu = \eta_{\mu\alpha} v^\alpha; \quad \eta^\dagger: \mathbb{M}^* \rightarrow \mathbb{M} \quad v^\mu = \eta^{\mu\alpha} v_\alpha$$

Tensor contraction

$$T_{\mu\nu}{}^\alpha \eta^\mu{}_\alpha = T_{\alpha\nu}{}^\alpha = B_\nu \quad T_{\mu\nu}{}^\alpha \text{ rank}(1/2) \text{ tensor} \rightarrow \text{rank}(1)$$

Trace of a tensor

$$T^\mu{}_\mu = T^0{}_0 + T^1{}_1 + \dots + T^m{}_m = T \in \mathbb{R} \quad \text{eg. } \eta^\mu{}_\mu = \delta^\mu{}_\mu = 4$$

Anti-symmetry

$$A_{\mu\nu} = -A_{\nu\mu}, \quad A_{\mu\nu} S^{\mu\nu} = 0 \quad \text{with } S \text{ symmetric}$$

- Note: we used η but this formalism is valid for any arbitrary metric g ! in a given field $g \neq \eta$!

The metric is "hidden" in many places!

• Proper time $\tau = \frac{1}{c} \sqrt{-ds^2} = \frac{1}{c} \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu} = \frac{1}{c} \sqrt{-\eta(d\bar{x}, d\bar{x})}$

↑

• Another example? It is even in the dear old Kinetic energy ...

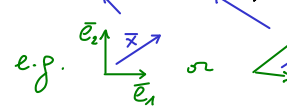
$$T = \frac{1}{2} m \bar{v}^2 = \frac{1}{2} m \bar{v} \cdot \bar{v} = \frac{1}{2} m \delta_{ij} v^i v^j \quad \bar{v} \in \mathbb{R}^3$$

↑ $\delta_{ii} = 1, \delta_{ij} = 0 \text{ for } i \neq j$ (Euclidean 3D space + cartesian system)

Note: careful with vectors...

- Vectors do not necessarily have coefficients !!! $\bar{x}$
- They have coefficients once you specify a basis set $\{\bar{e}_\mu\}$ for their vector space

$\bar{x} = x^\mu \bar{e}_\mu = x^{\mu'} \bar{e}_{\mu'}$ 4-vectors as linear combination of basis

e.g.  different $\{\bar{e}_\mu\}$ but $\bar{x}$ is $\bar{x}$

$\{\bar{e}_\mu\}$ basis set

↑
some linearly independent vectors

$\eta_{\mu\nu} = \eta(\bar{e}_\mu, \bar{e}_\nu) = \bar{e}_\mu \cdot \bar{e}_\nu$ η defines the scalar product

$\bar{e}_0 \cdot \bar{e}_0 = -1 \quad \bar{e}_i \cdot \bar{e}_i = 1 \quad \bar{e}_i \cdot \bar{e}_j = 0 \quad i \neq j$

$\bar{e}_\mu, \bar{e}_{\mu'}$ basis of S and S' frame

← Cartesian frame

$\bar{e}_0 = (1, 0, 0, 0)^T, \bar{e}_1 = (0, 1, 0, 0)^T, \dots$
 $ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$

Minkowski metric η can also be not diagonal. Example with $\neq$ basis:

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 = \underbrace{(-c dt + dx)}_{\equiv du} \underbrace{(c dt + dx)}_{\equiv dv} + dy^2 + dz^2 \equiv du dv + dy^2 + dz^2$$

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\eta = \begin{pmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Other coordinate frames: polar, Kruskal, Eddington-Finkelstein, ...

This is just linear algebra!

Frame transformations

We deal with transformations between inertial frames

$$x^\mu \rightarrow x^{\mu'} \quad \begin{cases} * \left\{ \begin{array}{l} \frac{dx^\mu}{d\lambda} = \text{const.} \quad \frac{dx^{\mu'}}{d\lambda} = \text{const.} \quad (\text{shift of coordinates through a point}) \\ ** \left\{ \begin{array}{l} \frac{d^2 x^\mu}{d\lambda^2} = 0 \quad \frac{d^2 x^{\mu'}}{d\lambda^2} = 0 \quad (\text{no acceleration}) \end{array} \right. \end{array} \right. \quad \lambda: \text{affine parameter } x^\mu(\lambda)$$

Relation between frames S, S'

$$x^{\mu'} = x^{\mu'}(x^\mu(\lambda)) \quad \frac{dx^{\mu'}}{d\lambda} = \frac{\delta x^{\mu'}}{\delta x^\mu} \frac{dx^\mu}{d\lambda} \quad J_{\mu'}^\mu = \frac{\delta x^{\mu'}}{\delta x^\mu} \quad \text{Jacobian of the transformation}$$

↑ transformations law
↑ transformed component

$$** \stackrel{!}{=} 0 \quad \frac{d^2 x^{\mu'}}{d\lambda^2} = \frac{d}{d\lambda} \left(\frac{dx^{\mu'}}{d\lambda} \right) = \frac{d}{d\lambda} \left(\frac{\delta x^{\mu'}}{\delta x^\mu} \frac{dx^\mu}{d\lambda} \right) = \frac{\delta^2 x^{\mu'}}{\delta x^\nu \delta x^\mu} \frac{dx^\nu}{d\lambda} \frac{dx^\mu}{d\lambda} + \frac{\delta x^{\mu'}}{\delta x^\mu} \frac{d^2 x^\mu}{d\lambda^2} \stackrel{!}{=} 0 \Rightarrow \boxed{\frac{\delta^2 x^{\mu'}}{\delta x^\nu \delta x^\mu} \stackrel{!}{=} 0}$$

* constant
** $\stackrel{!}{=} 0$ (inertial frame)

$$\text{Solve for } x^{\mu'}: \quad 0 = \int \frac{\delta^2 x^{\mu'}}{\delta x^\nu \delta x^\mu} x^\mu dx^\nu = \frac{\delta x^{\mu'}}{\delta x^\mu} x^\mu - \int \underbrace{\frac{\delta x^{\mu'}}{\delta x^\mu} \frac{\delta x^\mu}{\delta x^\nu}}_{= \frac{\delta x^{\mu'}}{\delta x^\nu}} dx^\nu = J_{\mu'}^\mu x^\mu - x^{\mu'} + \underbrace{a^{\mu'}}_{\text{integration constant}} = 0$$

This implies a linear transformation of vectors between frames

$$x^{\mu'} = J_{\nu}^{\mu'} x^\nu + a^{\mu'} \quad \bar{x}' = J \bar{x}$$

origin shift (irrelevant) (matrix)

Distance between points invariant under coord. transformation

$$\begin{aligned} \rightarrow \text{i.e. preserves scalar product} & \quad \boxed{\langle \bar{u}, \bar{v} \rangle = \langle \bar{u}', \bar{v}' \rangle} \quad (1) \\ \rightarrow \text{i.e. metric is invariant} & \quad \boxed{\eta = J^T \eta J} \quad (2) \quad \text{i.e. } J \text{ is an isometry} \end{aligned} \quad \Rightarrow \text{Preserves homogeneity}$$

$\eta = \eta'$ is the necessary and sufficient condition for $J_{\nu}^{\mu'} \equiv \frac{\delta x^{\mu'}}{\delta x^\nu}$ to be a Lorentz transformation

Conditions (1) and (2) are equivalent:

$$\begin{aligned} d\bar{s}^2 = \eta(d\bar{x}, d\bar{x}) & \stackrel{!}{=} \eta(d\bar{x}', d\bar{x}') \quad (1) \\ \downarrow \\ \bar{x}^T \eta \bar{x} & \stackrel{!}{=} \bar{x}'^T \eta \bar{x}' = \bar{x}^T J^T \eta J \bar{x} \Rightarrow (2) \quad \eta = J^T \eta J \quad \Leftrightarrow \quad \eta_{\mu\nu} = J_{\mu}^{\alpha'} J_{\nu}^{\beta'} \eta_{\alpha'\beta'} \\ & \quad \text{or } \eta_{\alpha'\beta'} = J_{\alpha'}^{\mu} J_{\beta'}^{\nu} \eta_{\mu\nu} \end{aligned}$$

Coefficients of the Lorentz transformation:

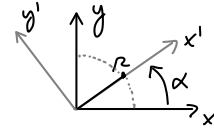
$$(J_{\nu}^{\mu'}) \stackrel{!}{=} (\Lambda_{\nu}^{\mu'}) = \left(\frac{\delta x^{\mu'}}{\delta x^\nu} \right) = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} ct' \\ x' \end{pmatrix} = \Lambda \begin{pmatrix} ct \\ x \end{pmatrix} \quad \text{for "normal configuration"}$$

Lorentz invariant quantities

Are quantities that do not change under Lorentz transformations
i.e. they have the same value $\forall$ frame. How can it be?!

A familiar example: rotations in 3D

e.g. in 2D $r^2 = x^2 + y^2 = \delta_{ij} x^i x^j$ Invariant under rotations



$$= (r \cos \alpha)^2 + (r \sin \alpha)^2 \quad r^2 \geq 0 \quad \text{corresponds to the statement } \cos^2 \alpha + \sin^2 \alpha = 1 \quad \forall \alpha \in \mathbb{R}$$

Likewise, for Lorentz transforms: define a "rotation angle" Υ

• Υ = rapidity parameter (Sort of polar coord.)

$$\cosh \Upsilon \equiv \gamma \in [1, \infty]$$

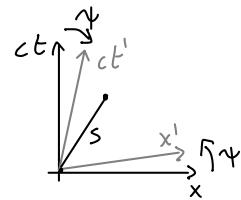
$$\sinh \Upsilon \equiv \beta \gamma \in [-\infty, \infty]$$

$$\tanh \Upsilon = \frac{\beta \gamma}{\gamma} = \beta$$

$$\cosh^2 \Upsilon - \sinh^2 \Upsilon = \gamma^2 - \beta^2 \gamma^2 = \gamma^2 (1 - \beta^2) = 1$$

$$s^2 = -(dx^0)^2 + (dx^1)^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

$$= -(\underbrace{s \sinh \Upsilon}_{\approx 1})^2 + (\underbrace{s \cosh \Upsilon}_{\approx 1})^2 = s^2 (\underbrace{\cosh^2 \Upsilon - \sinh^2 \Upsilon}_{=1})$$



• Lorentz transformations $\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh \Upsilon & \sinh \Upsilon \\ \sinh \Upsilon & \cosh \Upsilon \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} \sim \text{hyperbolic rotation : boost}$

• Λ form the so called Lorentz group (see next page)
the invariance is an intrinsic property of this group
not just a property of $\sinh, \cosh$!

The Lorentz group

The Lorentz transformations form a non Abelian Lie group

$$SO(3,1, \mathbb{R}) = \{ \Lambda \in \mathcal{M}(4, \mathbb{R}) \mid \langle \vec{u}, \vec{v} \rangle = \langle \Lambda \vec{u}, \Lambda \vec{v} \rangle \quad \forall \vec{u}, \vec{v} \in \mathbb{M} \}$$

↑ orthogonal $\Lambda^T \Lambda = I_4$ ↑ 4x4 real matrixes

Special: $\det(\Lambda) = 1$ in general: $\det(\Lambda) = \pm 1$ (-) improper (+) proper we consider this subgroup

In fact:

- the product of 2 Lorentz transformations is another Lorentz transf. (addition)
 - " " is associative $(\Lambda^\alpha_\beta \Lambda^\beta_\gamma) \Lambda^\gamma_\sigma = \Lambda^\alpha_\beta (\Lambda^\beta_\gamma \Lambda^\gamma_\sigma)$
 - there is an identity transformation $\Lambda^\mu_\nu = \delta^\mu_\nu$ ($v=0$)
 - " " " inverse $\forall \Lambda : (\Lambda^{-1})^\mu_\nu = \Lambda_\nu^\mu$
- } group

Moreover:

- The group is non Abelian i.e. Λ do not commute (for rotations) $\Lambda(\vec{v}_1) \Lambda(\vec{v}_2) \neq \Lambda(\vec{v}_2) \Lambda(\vec{v}_1)$
- " " is a Lie group i.e. its elements $\Lambda(\vec{v})$ are "functions of continuous parameters v_i or φ_i
finite dimensional differentiable (i.e. smooth) manifold
(= it is a continuous differentiable group)
- " " is a Poincaré group i.e. it has 10 parameters:
6 free parameters (1): 3 boosts (x^i, x^0 planes), 3 rotations (x^i, x^j $i \neq j$ planes) $\dim[O(n)] = \frac{n(n-1)}{2} = 6$ (with $n=4$)
4 translations (2): shift in time and space
$$x^{\mu'} = \underbrace{\Lambda^{\mu'}_\nu}_{(1)} x^\nu + \underbrace{a^{\mu'}}_{(2)}$$

Proves:

$$\eta_{\mu\nu} = \Lambda^{\alpha'}_\mu \Lambda^{\beta'}_\nu \eta_{\alpha'\beta'} = \Lambda^{\beta'}_\mu \Lambda^{\alpha'}_\nu \eta_{\alpha'\beta'} = \Lambda^{\alpha'}_\mu \Lambda^{\beta'}_\nu \eta_{\alpha'\beta'} \Rightarrow \Lambda^{\alpha'}_\mu \Lambda^{\beta'}_\nu = \delta^{\alpha\beta} \Rightarrow \Lambda^{\alpha'}_\mu \text{ is the inverse of } \Lambda^{\beta'}_\alpha$$

Take infinitesimal Lorentz transf. $\Lambda^\mu_\nu = \delta^\mu_\nu + \delta\omega^\mu_\nu$ ($|\delta\omega^\mu_\nu| \ll 1$) because of $\eta_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta = \eta_{\alpha\beta}$

$$\eta_{\mu\nu} (\delta^\mu_\alpha + \delta\omega^\mu_\alpha) (\delta^\nu_\beta + \delta\omega^\nu_\beta) = (\eta_{\mu\nu} + \delta\omega_{\nu\alpha}) (\delta^\nu_\beta + \delta\omega^\nu_\beta) = \eta_{\alpha\beta} + \delta\omega_{\beta\alpha} + \delta\omega_{\alpha\beta} + \underbrace{\delta\omega_{\alpha\beta} \delta\omega^\nu_\beta}_{2^{\text{nd order}}}$$

$$\Rightarrow \delta\omega_{\beta\alpha} = -\delta\omega_{\alpha\beta} \quad \text{i.e. antisymmetric} \Rightarrow 6 \text{ free parameters}$$

3 rotations $\delta\omega_{ij} = -\epsilon_{ijk} M^k \delta\theta$ rotation of angle $\delta\theta$ around unit vector $\vec{m}$

3 boosts $\delta\omega_{i0} = m_i \delta\gamma$ boost with rapidity $\delta\gamma$ along direction $\vec{m}$

Infinitesimal Lorentz transf. are proper

Subgroups:

- because of $(\Lambda^{-1})^\mu{}_\rho = \Lambda_\rho{}^\mu$: in general $\boxed{\det(\Lambda) = \pm 1}$
 - (-) improper Lorentz transf.
 - (+) proper Lorentz transf.

we consider this subgroup ↗
- because of $\eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta}$: $\eta_{\mu\nu} \Lambda^\mu{}_0 \Lambda^\nu{}_0 = -\Lambda_0{}^0 \Lambda_0{}^0 + \delta_{ij} \Lambda^i{}_0 \Lambda^j{}_0 = \eta_{00} = -1$

$$\Rightarrow (\Lambda_0{}^0)^2 - \Lambda^i{}_0 \Lambda^i{}_0 = 1 \text{ i.e. } \boxed{\Lambda_0{}^0 \geq +1} \leftarrow \text{orthochronous Lorentz group}$$

$$\boxed{\Lambda_0{}^0 \leq -1}$$

? symmetry? ↓?
 $\eta_{\mu\kappa} \Lambda^\mu{}_0 \Lambda^\kappa{}^\alpha = \eta_{00} \Lambda^0{}^\alpha + \eta_{ij} \Lambda^i{}_0 \Lambda^j{}^\alpha$
- infinitesimal Lorentz transf. are proper and orthochronous
- Generally, a theory is said to be Lorentz invariant for proper orthochronous sub group only
 - $\Rightarrow$ The Lorentz sub group we are interested in can be covered by infinitesimal transf.
- 2 discrete transf. to take us out of these sub groups
 - parity reversal: $P^\mu{}_\nu = (P^{-1})^\mu{}_\nu = \text{diag}(1, -1, -1, -1)$ orthochronous and improper
 - time reversal: $T^\mu{}_\nu = (T^{-1})^\mu{}_\nu = \text{diag}(-1, 1, 1, 1)$ non orthochronous and improper

• What is a groups?

Set of elements $\{G_i\}$ with a connection $*$ between elements $(G, *)$ with these properties:

$*$	connection between elements	: e.g. $(\mathbb{R}, +)$ $1_G = 1$ $(\mathbb{R}, \cdot)$ $1_G = 0$ $(\mathbb{R}^n \times \mathbb{R}^n, \cdot)$ $1_G = I^n$
$a * b = c$	addition, $\forall a, b \in G \Rightarrow c \in G$	
$(a * b) * c = a * (b * c)$	associativity, $\forall a, b, c \in G$	
$e * a = a * e = a$	$\exists$ of identity element e , $\forall a \in G$	
$a * b = b * a = e$	$\exists$ of inverse, $\forall a, b \in G$	
<u>extra</u> $a * b = b * a$	commutativity, $\forall a, b \in G \Rightarrow$ <u>Abelian group</u> , e.g. $GL(n)$ is not abelian Rotations	

• What is a Lie group?

Continuous group $\{G(\alpha_1, \dots, \alpha_m)\}$ i.e. Elements are "functions" of α_i continuous parameters
 Have finite dimensional differentiable (i.e. smooth) manifold

Relativistic mechanics

– Behaviour of particles in the space-time

– 4-vector : $(x^\mu) = (x^0, x^1, x^2, x^3) = (ct, x^1, x^2, x^3)$ not invariant

– World-line : $x^\mu(\tau)$ trajectory of particle in space-time τ = proper time

– 4-velocity : $u^\mu \equiv \frac{dx^\mu}{d\tau} = \gamma \frac{dx^\mu}{dt}$ $(u^\mu) = \gamma \left(c \frac{dt}{dt}, \frac{dx^1}{dt}, \frac{dx^2}{dt}, \frac{dx^3}{dt} \right) = \gamma(c, \vec{v}) = c\gamma(1, \vec{\beta})$

– note at which coordinates pass by an observer

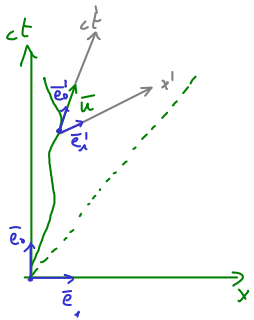
– not invariant

– $\langle \bar{u}, \bar{u} \rangle = u_\mu u^\mu = \eta_{\mu\nu} u^\nu u^\mu = c^2 \gamma^2 (-1 + \vec{\beta}^2) = -c^2 < 0$ invariant
(this is because of ds^2 : $-c^2 d\tau^2 = dx_\mu dx^\mu \Rightarrow -c^2 = \frac{dx_\mu}{d\tau} \frac{dx^\mu}{d\tau} = u_\mu u^\mu$)

– time-like $\Rightarrow$ particles can only move inside the light cone

– $\bar{u}$ is tangent to the world line because it is a multiple of $d\bar{x}$: $\bar{u} = \frac{d\bar{x}}{d\tau}$ scalar

– In particle rest frame $\bar{u}' = (c, 0, 0, 0)^T \Rightarrow \bar{u} = c \bar{e}_0$



– 4-acceleration : $a^\mu \equiv \frac{du^\mu}{d\tau}$ $(a^\mu) = \frac{d}{d\tau} [\gamma(c, \vec{v})] = \gamma \frac{d}{dt} [\gamma(c, \vec{v})] = \gamma(\dot{\gamma}c, \dot{\gamma}\vec{v} + \gamma\dot{\vec{v}})$

– note : $\frac{d}{d\tau} (u^\mu u_\mu) = \frac{du^\mu}{d\tau} u_\mu + u^\mu \frac{du_\mu}{d\tau} = \frac{du^\mu}{d\tau} u_\mu + u_\mu \frac{du^\mu}{d\tau} = 2 u_\mu a^\mu = 0$

$\Rightarrow \bar{u} \perp \bar{a}$ always! (orthogonal in 4-D)

– In instantaneous rest frame : $\gamma=1, \dot{\gamma}=0, (a^\mu) = (0, \vec{a})$

– 4-momentum : $p^\mu \equiv m_0 u^\mu$

• $(p^\mu) = \gamma(m_0 c, m_0 \vec{v})$ $\left\{ \begin{array}{l} (1) \gamma m_0 c = m_0 c (1 - \beta^2)^{-1/2} \approx m_0 c (1 + \frac{1}{2} \beta^2 + \dots) = m_0 c + \frac{1}{2} m_0 c \frac{v^2}{c^2} = \frac{1}{c} (m_0 c^2 + \frac{1}{2} m_0 v^2) \stackrel{!}{=} \frac{E}{c} \\ (2) \vec{p} = \gamma m_0 \vec{v} \end{array} \right.$
 $= \gamma \left(\frac{E_0}{c}, m_0 \vec{v} \right) = \left(\frac{E}{c}, \vec{p} \right)$ $\leftarrow$ Energy-momentum (one entity!)
 Energy and momentum are components of the same entity

• $\langle \bar{p}, \bar{p} \rangle = -\frac{E^2}{c^2} + \vec{p}^2$
 $\langle \bar{p}, \bar{p} \rangle = p_\mu p^\mu = m_0^2 u_\mu u^\mu = -m_0^2 c^2$ $\left\} \begin{array}{l} -\frac{E^2}{c^2} + \vec{p}^2 = -m_0^2 c^2 \\ E^2 = m_0^2 c^4 + c^2 \vec{p}^2 \end{array} \right.$ at rest $E = m_0 c^2$
 m_0 = rest mass

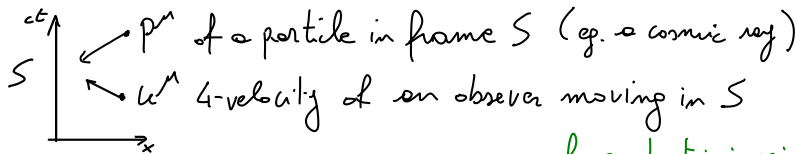
• $\frac{E}{c} = \gamma m_0 c, \vec{p} = \gamma m_0 \vec{v} \Rightarrow \vec{p} = \frac{E}{c^2} \vec{v}$

• Constraint on velocity of massive particles

$$p_\mu p^\mu = -p^2 + p^2 = -m_0^2 c^2 \Rightarrow p^\mu = (p^0 - m_0^2 c^2)^{1/2} \quad \frac{p^1}{p^0} = \left(1 - \frac{m_0^2 c^2}{p^2}\right)^{1/2} \rightarrow 1 \text{ (but not 1!)} \text{ for } p^0 = \frac{E}{c} \rightarrow \infty$$

$\Rightarrow$ No matter how much energy we give to the particle, $v < c$!

• A convenient expression: energy of a particle measured by a moving observer



scalar product is invariant

$$p_\mu u^\mu = p'_\mu u'^\mu = \left(\frac{E'}{c}, \vec{p}'\right) \left(\frac{c}{0}\right) = -\frac{E'}{c} \cdot c + \vec{p}' \cdot \vec{0} = -E' \Rightarrow E' = -p_\mu u^\mu$$

- 4-force: $g^\mu \equiv \frac{dp^\mu}{d\tau}$ $(g^\mu) = \gamma \frac{d}{dt} \left(\frac{E}{c}, \vec{p}\right) = \gamma \left(\frac{\vec{F} \cdot \vec{v}}{c}, \vec{F}\right)$

① $dE = \vec{F} \cdot d\vec{x} = \vec{F} \cdot \vec{v} dt \Rightarrow \dot{E} = \vec{F} \cdot \vec{v}$ work

② $\dot{\vec{p}} = \vec{F}$ 3-force

remember: a force is a change in momentum not $\vec{F} = m\vec{a}$

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v}) = m\dot{\vec{v}} \quad \uparrow \quad m = \text{const.}$$

• Change in rest mass ?!

product rule (let m free)

$$u^\mu g_\mu = u^\mu \frac{dp_\mu}{d\tau} = u^\mu \frac{d}{d\tau}(m u_\mu) = u^\mu \left(\frac{dm}{d\tau} u_\mu + m \frac{du_\mu}{d\tau} \right) = \underbrace{u^\mu u_\mu}_{=-c^2} \frac{dm}{d\tau} + \underbrace{m u^\mu \frac{du_\mu}{d\tau}}_{=0} = -c^2 \frac{dm}{d\tau}$$

$\Rightarrow$ A force may change rest mass

$$\frac{dm}{d\tau} = -\frac{u^\mu g_\mu}{c^2} \quad \text{if } u^\mu g_\mu \neq 0 !$$

• If $u^\mu g_\mu = 0 \Rightarrow \frac{dm}{d\tau} = 0$

g^μ called pure force $\gamma \left(\frac{\vec{F} \cdot \vec{v}}{c}, \vec{F}\right)$

$\downarrow$
acts on 3-acceleration
but not on rest energy

Variational approach

– Free particle : no external forces

– Action : $S \in \mathbb{R}$ must be Lorentz invariant (not dependent on motion of observers)

$\Rightarrow S$ must depend on Lorentz scalar: τ is the only one characterizing the particle

$$S = \int L dt = \alpha \int d\tau = \alpha \int \gamma^{-1} dt$$

non-relat. limit
 $\beta \ll 1$

no external forces
 $\downarrow$

$$\text{Find } \alpha: L = \alpha \gamma^{-1} = \alpha (1 - \beta^2)^{1/2} = \alpha (1 - \frac{1}{2} \beta^2 + \dots) \simeq \alpha - \frac{1}{2} \alpha \frac{\vec{v}^2}{c^2} \stackrel{!}{=} T = \frac{1}{2} m \vec{v}^2 \Rightarrow \alpha = -m_0 c^2$$

$$L = -m_0 c^2 \gamma^{-1} = -m_0 c^2 (1 - \beta^2)^{1/2} \quad \leftarrow \text{Lagrangian of a free particle}$$

• Note: S has a clear physical meaning in S.R. : interval of proper time along world line

$L \rightarrow L' = aL + b$ gives same eq. of motion ($\delta S = 0$) a = time rescaling, b = time shift

– Conjugate momentum :

$$\vec{p} = \frac{\delta L}{\delta \vec{v}} = \cancel{+m_0 \gamma} \frac{1}{2} (1 - \beta^2)^{-1/2} \cancel{\gamma} \frac{1}{c^2} \vec{v} = \gamma m_0 \vec{v} \quad \boxed{\vec{p} = \gamma m_0 \vec{v}} \text{ as seen before}$$

– Hamiltonian :

$$H = \vec{v} \cdot \vec{p} - L = m_0 \gamma \vec{v}^2 + m_0 c^2 \gamma^{-1} = \gamma m_0 c^2 (1 - \beta^2 + \beta^2) = \gamma m_0 c^2 \quad \boxed{E = \gamma m_0 c^2}$$

$$(P^\mu) = \left(\frac{E}{c}, \vec{p} \right)$$

elements of a 4-vector as we have seen!

– Equation of motion (Euler-Lagrange eq.) :

$$\begin{aligned} \frac{d}{dt} \frac{\delta L}{\delta \vec{v}} - \frac{\delta L}{\delta \vec{x}} &= \frac{d}{dt} \left(\gamma m_0 \vec{v} \right) = m_0 \left[\dot{\gamma} \vec{v} + \gamma \dot{\vec{v}} \right] \quad \dot{\gamma} = \frac{1}{2} (1 - \beta^2)^{-3/2} \cancel{\gamma} \beta \dot{\beta} = \gamma^3 \frac{\vec{v} \cdot \dot{\vec{v}}}{c^2} \\ &= m_0 \left(\gamma^3 \frac{\vec{v} \cdot \dot{\vec{v}}}{c^2} \vec{v} + \gamma \dot{\vec{v}} \right) \\ &= m_0 \dot{\vec{v}} \left(\underbrace{\gamma^3 \frac{\vec{v}^2}{c^2} + \gamma}_{>0} \right) \stackrel{!}{=} 0 \Rightarrow \boxed{\dot{\vec{v}} = 0} \text{ no acceleration as expected} \\ &\quad \downarrow \vec{x} = \vec{a}t + \vec{b} \text{ move along straight line} \end{aligned}$$

– Equation of motion (least action principle 4D):

$$S = -m_0 c^2 \int d\tau = -m_0 c^2 \int \frac{1}{c} \left(-\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right)^{1/2} d\tau = -m_0 c \int \left(-\eta_{\mu\nu} u^\mu u^\nu \right)^{1/2} d\tau$$

$$\begin{aligned} \delta S &= -m_0 c \int_A^B \frac{1}{2} \left(-u_\nu u^\nu \right)^{-1/2} \delta \left(-u_\nu u^\nu \right) d\tau = m \int_A^B u_\nu \frac{d}{d\tau} \delta x^\nu d\tau = m \int_A^B u_\nu \frac{d}{d\tau} \delta x^\nu d\tau \quad \leftarrow \text{by part} \\ &= m \left[u_\nu \delta x^\nu \Big|_A^B - \int_A^B \frac{du_\nu}{d\tau} \delta x^\nu d\tau \right] \stackrel{!}{=} 0 \quad \forall \delta x^\nu \Rightarrow \text{4-velocity} \quad \boxed{\frac{du^\nu}{d\tau} = 0} \end{aligned}$$

$$\Rightarrow \frac{du^\nu}{d\tau} = \frac{d^2 x^\nu}{d\tau^2} = 0 \Rightarrow \boxed{x^\nu = \alpha^\nu \tau + \beta^\nu} \Rightarrow \begin{cases} \text{no acceleration, motion along a straight line} \\ \text{no change in particle's energy } m \frac{du^0}{d\tau} = \frac{dP^0}{d\tau} = 0 \end{cases}$$

Photons

• massive particles $ds^2 = c^2 d\tau^2 < 0$ ($d\tau \neq 0!$) $\Rightarrow p^\mu = m \frac{dx^\mu}{d\tau}$ is well defined ✓

• photons $ds^2 = 0$ $-c^2 d\tau^2 = \downarrow ds^2 = 0$ proper time on the light-cone is identically zero!
 $\Rightarrow u^\mu = \frac{dx^\mu}{d\tau}$ $\leftarrow \tau=0!$ 4-velocity is not well defined!

We must use another affine parameter (not τ) $\rightarrow \lambda$

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = \eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} d\lambda^2 = \eta_{\mu\nu} K^\mu K^\nu d\lambda^2 \quad K^\mu \equiv \frac{dx^\mu}{d\lambda} \quad dx^\mu = K^\mu d\lambda$$

(sort of 4-velocity) (shift)

We can use K^μ to define the momentum of photons

$(p^\mu) = (\frac{E}{c}, \vec{p})$ for photons as they obs have E and $\vec{p}$, $p^\mu = K^\mu$

1) They move along null lines: $p_\mu p^\mu = -E^2 + \vec{p}^2 = 0$ $E^2 = c^2 \vec{p}^2 \Rightarrow \boxed{m=0!}, \boxed{E=cp}$ $E^2 = c^2 \vec{p}^2 + m^2 c^4$

2) Energy is quantized: $E = h\nu = \hbar\omega$ $p = \frac{h\nu}{c} = \frac{\hbar\omega}{c} = \frac{h}{\lambda}$ $\omega = \frac{v}{2\pi}$ $\lambda = \frac{c}{v}$

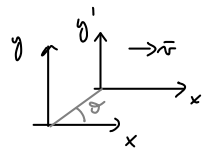
$$(p^\mu) = (\frac{E}{c}, p\vec{e}) = \frac{E}{c}(1, \vec{e}) = \frac{\hbar\omega}{c}(1, \vec{e}) = \hbar(K^\mu) \Rightarrow \boxed{(K^\mu) = \frac{\omega}{c}(1, \vec{e}) = \frac{2\pi}{\lambda}(1, \vec{e})} = (\frac{2\pi}{\lambda}, \vec{k})$$

"4-frequency"

Lorentz transform of 4-frequency:

$$(K^\mu) = (\frac{2\pi}{\lambda}, \vec{k}) = \frac{2\pi}{\lambda}(1, \cos\theta, \sin\theta, 0)$$

$$(K'^\mu) = (\frac{2\pi}{\lambda'}, \vec{k}') = \frac{2\pi}{\lambda'}(1, \cos\theta', \sin\theta', 0)$$



$$K'^\mu = \Lambda^\mu{}_\alpha K^\alpha \quad (\mu=0) \text{ time: } \frac{\lambda}{\lambda'} = \gamma(1 - \beta \cos\theta) \quad \leftarrow \text{Doppler effect}$$

$$(\mu=i) \text{ space: } \tan\theta' = \frac{\tan\theta}{\gamma(1 - \beta \cos\theta)} \quad \leftarrow \text{Aberration of light}$$

Example:

$$\mu=0 \quad K^0 = \Lambda^0{}_\alpha K^\alpha = \Lambda^0{}_0 K^0 + \Lambda^0{}_1 K^1 + \Lambda^0{}_2 K^2 + \Lambda^0{}_3 K^3$$

$$= \gamma K^0 - \beta\gamma K^1 + 0 + 0 = \frac{2\pi}{\lambda}(\gamma - \beta\gamma \cos\theta) \quad \frac{2\pi}{\lambda'} = \frac{2\pi}{\lambda} \gamma(1 - \beta \cos\theta)$$

Doppler shift

Appendix

Decay of particles

Example: body of mass M splits in two parts of mass m_1, m_2

$\Rightarrow M = m_1 + m_2$? No!

• In rest frame of body : $E = Mc^2$

• Energy conservation : $Mc^2 = E_1 + E_2$ (1) split $\Rightarrow m_1, m_2$ move apart $\Rightarrow |\vec{p}_i| > 0$
 $\Downarrow$ $\Rightarrow E_i^2 = m_i^2 c^4 + c^2 \vec{p}_i^2 > m_i^2 c^4$
 $M > m_1 + m_2$!

Spontaneous decay is possible

- if $M < m_1 + m_2 \Rightarrow$ decay is not possible (system is stable)
 $\Rightarrow$ no spontaneous decay (or energy conservation would be violated)
 $\Rightarrow$ to have decay, you need to give energy to the body
 of an amount $\Delta E = m_1 + m_2 - M$

• Momentum conservation : $0 = \vec{p}_1 + \vec{p}_2 \Rightarrow \vec{p}_1^2 = \vec{p}_2^2 \quad c^2 \vec{p}_i^2 = E_i^2 - m_i^2 c^4$
 $\Downarrow$
 energies of the components E_1, E_2
 $E_1^2 - m_1^2 c^4 = E_2^2 - m_2^2 c^4 \quad \leftarrow \text{Plug (1)}$
 $\Rightarrow E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M} c^2 \quad E_2 = \frac{M^2 + m_2^2 - m_1^2}{2M} c^2$

Inverse process

at rest (lab)
 $\vec{v}_1$
 $m_1 \rightarrow m_2 \Rightarrow M$

$$E = E_1 + E_2 = E_1 + m_2 c^2$$

$$\vec{p} = \vec{p}_1 + \vec{p}_2 = \vec{p}_1$$

$\Rightarrow$ Final velocity: $\vec{p} = \frac{E \vec{v}}{c^2} \Rightarrow \vec{v} = \frac{c^2 \vec{p}}{E} = \frac{c^2 \vec{p}_1}{E_1 + m_2 c^2}$ velocity of the merged body in the lab frame

$\Rightarrow$ Final mass M : $E^2 = M^2 c^4 + c^2 \vec{p}^2$
 $c^4 M^2 = E^2 - c^2 \vec{p}^2$
 $= (E_1 + m_2 c^2)^2 - (E_1^2 - m_1^2 c^4)$
 $= \cancel{E_1^2} + m_2^2 c^4 + 2E_1 m_2 c^2 - \cancel{E_1^2} + m_1^2 c^4$
 $= c^4 \left(m_1^2 + m_2^2 + \frac{2E_1 m_1}{c^2} \right) \Rightarrow$

$$M^2 = m_1^2 + m_2^2 + \frac{2E_1 m_1}{c^2}$$

Dispersion relation for massive particles

It relates frequency (energy) with wavenumber (momentum)

In quantum mechanics: $E = \hbar \omega$ $\vec{P} = \hbar \vec{k}$

Relativistic quantum mechanics: $E^2 = m^2 c^4 + c^2 \vec{P}^2 \rightarrow \omega^2 = c^2 \vec{k}^2 + \left(\frac{m c^2}{\hbar}\right)^2$

Wave frequency, ω , depends on wavevector and mass

Phase velocity: $v_{ph} \equiv \frac{\omega}{k}$ speed of wave peaks (constant phase points move)

$$= \frac{E}{P} = \frac{\sqrt{m^2 c^4 + c^2 \vec{P}^2}}{P} = c \sqrt{1 + \left(\frac{m c}{P}\right)^2}$$

it can exceed speed of light (for $m > 0$)
OK, it does carry information

Group velocity: $v_g \equiv \frac{d\omega}{dk}$ speed of wave packets (energy, information)

derivative of the dispersion relation, i.e. how energy changes with wavenumber

$$= \frac{dE}{dP} = \frac{P c^2}{E} < c$$

which is the velocity of particles in special relativity

"Backward", using their definitions

$$v_{ph} \cdot v_g = \frac{E}{P} \frac{dE}{dP} = \frac{d(E^2)}{d(P^2)} = c^2 \quad d(E^2) = c^2 d(P^2) \quad \text{integrate} \Rightarrow E^2 = c^2 P^2 + \text{const}$$

↑ integration constant
= rest mass pops up

Energy-Momentum tensor

- Particle: point object, "•", at position x^μ characterized by $(p^\nu) = (\frac{E}{c}, \vec{p}) \rightarrow$ conservation
- Fields/Fluids: $q(x^\mu)$ a quantity at all x^μ " " " $T^{\nu\sigma}(x^\mu)$ energy-momentum tensor
 $\hookrightarrow$ eg. matter field (scalar), q^ν wind (vectorial), $q^{\nu\sigma}$ electro-magnetic field (tensorial)
- Description/Behaviour of a system

- Lagrangian, Action $S = \int L dt = \int \mathcal{L} dt dV = \frac{1}{c} \int \mathcal{L} d\Omega$ $d\Omega \equiv dx^0 dx^1 dx^2 dx^3$ ^{cdf}
- Euler-Lagrange eq. $\delta S = 0 \rightarrow \frac{\delta}{\delta x^\nu} \left(\frac{\delta \mathcal{L}}{\delta q^\mu_{,\nu}} \right) - \frac{\delta \mathcal{L}}{\delta q^\mu} = 0$ eq. of motion
evolution of the system

- Identify conserved quantities: $\delta \mathcal{L} = 0 = \mathcal{L}(x') - \mathcal{L}(x)$ corresponds to a symmetry of the system
(Noether theorem) infinitesimal translation $x'^\mu = x^\mu + \epsilon^\mu$ $\epsilon^\mu \ll 1$

$$\begin{aligned}
 q^\mu(x) &= q^\mu(x + \epsilon) \approx q^\mu(x) + \epsilon^\nu \partial_\nu q^\mu(x) & \delta q^\mu &= \epsilon^\nu \partial_\nu q^\mu & \delta \mathcal{L} &= \epsilon^\nu \partial_\nu \mathcal{L} \\
 \delta \mathcal{L} &= \frac{\delta \mathcal{L}}{\delta q^\mu} \delta q^\mu + \frac{\delta \mathcal{L}}{\delta q^\mu_{,\nu}} \delta q^\mu_{,\nu} = \frac{\delta}{\delta x^\nu} \left(\frac{\delta \mathcal{L}}{\delta q^\mu_{,\nu}} \right) \delta q^\mu + \frac{\delta \mathcal{L}}{\delta q^\mu_{,\nu}} \delta q^\mu_{,\nu} = \partial_\nu \left(\frac{\delta \mathcal{L}}{\delta q^\mu_{,\nu}} \right) \delta q^\mu \\
 \partial_\nu \left(\frac{\delta \mathcal{L}}{\delta q^\mu_{,\nu}} \right) \delta q^\mu - \delta \mathcal{L} &= 0 \Rightarrow \partial_\nu \left(\frac{\delta \mathcal{L}}{\delta q^\mu_{,\nu}} \delta q^\mu - \mathcal{L} \right) \epsilon^\nu = 0 & \text{no metric is needed} \\
 \delta q^\mu &= \epsilon^\sigma \partial_\sigma q^\mu & \delta \mathcal{L} &= \epsilon^\sigma \partial_\sigma \mathcal{L} = \partial_\nu (\epsilon^\sigma \delta^\nu_\sigma \mathcal{L}) \\
 & \uparrow & & \uparrow \\
 & \delta q^\mu_{,\nu} & & \delta q^\mu_{,\nu} \\
 & \uparrow & & \uparrow \\
 & \delta q^\mu_{,\nu} & & \delta q^\mu_{,\nu}
 \end{aligned}$$

$\equiv T^\nu_\sigma$ Energy-momentum tensor
 (the quantity which is conserved!)

$$T^{\nu\mu} = \frac{\delta \mathcal{L}}{\delta q^\mu_{,\nu}} \delta q^\mu - \eta^{\nu\mu} \mathcal{L} = \eta^{\sigma\mu} T^\nu_\sigma$$

$$\partial_\mu T^{\sigma\mu} = 0 \quad \text{Conservation laws}$$

- T refers to densities $\Rightarrow$ integrate over volume (space-like hypersurface)

$$P_\nu = \frac{1}{c} \int d^3x T^\sigma_\nu(x^0) \quad \text{4-momentum} \quad \partial_\nu P_\nu = \int d^3x \partial_\nu T^\nu_\nu(x^0) = 0 \quad \text{4-momentum conservation}$$

- Note: careful with signature, for $(-+++)$ $\times -1$ here and in Lagrangians
- Note: in general T is not diagonal but it can be diagonalized
- Note: $T^{\mu\nu}$ is not unique

$$\begin{aligned}
 \tilde{T}^{\mu\nu} &= T^{\mu\nu} + \partial_\sigma \psi^{\mu\nu\sigma} \quad \text{with } \psi^{\mu\nu\sigma} = -\psi^{\mu\sigma\nu} \quad \text{antisymmetric with respect to } \sigma\nu \\
 \partial_\mu \tilde{T}^{\mu\nu} &= 0 \quad \text{still satisfied because of the antisymmetry of } \psi: \quad \partial_\nu \partial_\sigma \psi^{\mu\nu\sigma} = 0 \\
 & \quad \text{Symmetric} \quad \text{antisymmetric}
 \end{aligned}$$

What is the meaning of $T^{\mu\nu}$?

Analogy to particles

$$p^\nu = (\mathcal{E}, \vec{p}) \quad 4\text{-momentum, particle} \quad c p^0 = E \text{ energy} \quad p^i = \frac{E v^i}{c^2} = \frac{s^i}{c^2} \quad [s^i] = \text{J m} \quad \text{energy flux}$$

$$p^\nu = \frac{1}{c} \int_V d^3x T^{\nu\mu} \quad 4\text{-momentum, volume element} \quad c p^0 = E \Rightarrow T^{00} = \mathcal{E} \quad \text{energy density}$$

convention $c^2 p^i = s^i \Rightarrow c T^{0i} = s^i$ " " " flux

Investigating $T^{\mu\nu}$ itself

$$\boxed{T^{00}} = \dot{\rho} \frac{\delta \mathcal{L}}{\delta \dot{\rho}} - \mathcal{L} = \mathcal{E} \quad \text{Hamiltonian density} \Rightarrow \underline{T^{00} = \text{energy density!}}$$

$$\boxed{T^{0i}} : \quad \delta_\nu T^{\nu\mu} = \delta_\nu T^{\nu\mu} + \delta_i T^{i\mu} = 0 \quad \text{is a "continuity" equation for the quantity } T^{\nu\mu} \text{ with } \mu \text{ given}$$

$$\underline{\mu=0}: \quad \frac{\delta T^{00}}{c \delta t} + \delta_i T^{0i} = 0, \quad \frac{T^{00}}{c} \equiv \mathcal{E} \Rightarrow \frac{\delta \mathcal{E}}{\delta t} + c \delta_i T^{0i} = 0 \quad \left. \vphantom{\frac{\delta \mathcal{E}}{\delta t}} \right\} \text{integrate over volume } V$$

$$\int_V \delta_t \mathcal{E} dV = -c \int_V \delta_i T^{0i} dV \quad \left. \vphantom{\int_V} \right\} \text{Gauss theorem}$$

$$\delta_t \int_V \mathcal{E} dV = -c \int_{\partial V} T^{0i} dA_i \quad \Rightarrow \quad c T^{0i} \equiv s^i \text{ energy density flux } [\text{J m} / \text{s}]$$

time variation of energy within volume V variation due to energy flux across the surface A_i enclosing V

$$\boxed{T^{ij}} \quad \underline{\mu=i}: \quad \frac{\delta T^{0i}}{c \delta t} + \delta_j T^{ij} = \frac{\delta (s^i)}{c \delta t} + \delta_j T^{ij} = 0 \Rightarrow T^{ij} \text{ is the density flux of } p^i \equiv \frac{s^i}{c^2} \quad \text{momentum density}$$

$\Rightarrow T^{ij} \equiv \sigma^{ij}$ momentum density flux!
i-th component along j-th direction

$$\Rightarrow (T^{\mu\nu}) = \begin{pmatrix} \mathcal{E} & s_x/c & s_y/c & s_z/c \\ \cdot & \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \cdot & \cdot & \sigma_{yy} & \sigma_{yz} \\ \cdot & \cdot & \cdot & \sigma_{zz} \end{pmatrix}$$

$\vec{s}$ = energy-density flux $\left[\frac{d\mathcal{E}}{dt dA} \right]$

σ_{ij} = stress tensor $\left[\frac{dP}{dt dA} \right]$ "pressure"

• What about G.R.?

- In a local inertial frame (free fall) the above holds careful with interpretation

- Strong equivalence principle: $\delta_\alpha \rightarrow \nabla_\alpha$
 $\eta_{\mu\nu} \rightarrow g_{\mu\nu} \Rightarrow \boxed{\nabla_\mu T^{\mu\nu} = 0}$

actually not just strong equiv. principle -- there is something more fundamental, more on that later

Energy momentum tensors of a perfect relativistic fluid

perfect = No heat conduction, No viscosity

System of many particles: e.g. gas, fluids

$T^{\mu\nu}$ in general:

T^{00}	energy density
T^{0i}	energy density flux / momentum density
T^{ij}	stresses: momentum density flux / pressure and shear
$\varepsilon = T_{\mu\nu} u^\mu u^\nu$	energy density measured by an observer moving with u^ν

Properties of fluids: ρ, p, u^μ fully characterize a perfect fluid (ensemble properties)

$T_{\text{int}} = \text{diag}(\rho c^2, p, p, p)$ fluid at rest (isotropy is expected)


Apply constraints to get result $T^{\mu\nu} = A u^\mu u^\nu + B g^{\mu\nu}$ most general symmetric rank-2 tensor built on the above points $A, B = ?$ at rest: $u = (c, 0, 0, 0)^T \Rightarrow T^{00} = A u^0 u^0 + B g^{00} = c^2 A + B \stackrel{!}{=} \rho c^2, T^{ii} = 0 - B \stackrel{!}{=} p \Rightarrow B = -p, A = \rho + \frac{p}{c^2}$

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu + p g^{\mu\nu}$$

$$T = \left(\rho + \frac{p}{c^2}\right) \bar{u} \otimes \bar{u} + p g^{-1} \quad \text{for } u^\mu = 0: T^{00} = \left(\rho + \frac{p}{c^2}\right) c^2 - \cancel{p} = \rho c^2 \quad T^{ii} = p \quad \checkmark$$

Non ideal fluid $T^{\mu\nu} = T^{\mu\nu}_{\text{ideal}} + q^\mu u^\nu + q^\nu u^\mu + \pi^{\mu\nu}$ q, π are directional $\Rightarrow q=0=\pi$ in cosmology

q^μ : heat flux (orthogonal to u^μ)

$\pi^{\mu\nu}$: viscous shear stress tensor ($\pi^{\mu\nu} = \pi^{\nu\mu}, T_{\mu\nu}(\pi) = 0$, orthogonal to u^μ)

Pressureless fluid (dust) $p=0$ $T^{\mu\nu} = \rho u^\mu u^\nu$ Dark matter always, baryons on large scales

Hydrodynamics "for free":

$\Rightarrow \nabla_\mu T^{\mu\nu} = 0$ hydrodynamic equations!

eq. of motion of fluid

For a closed system of equations
6 variables to constrain ($\varepsilon, \rho, p, \bar{u}$)

$\nabla_\mu T^{0\mu} = 0$	energy conservation
$\nabla_\mu T^{i\mu} = 0$ Euler's eq.	momentum conservation
$\nabla_\mu \rho = 0$ continuity eq.	matter conservation
$p = p(\rho, T)$ eq. of state	"fluid property"

Energy-momentum tensor of electromagnetic fields

$$\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} = -\frac{1}{16\pi} F_{\alpha\beta} F_{\gamma\delta} g^{\alpha\gamma} g^{\beta\delta} \Rightarrow T^{\mu\nu} = \frac{1}{4\pi} \left[-F^{\mu\alpha} F^\nu{}_\alpha + \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right]$$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ electro magnetic field $A^\mu = 4\text{-potential}$

General Relativity in a nutshell

What is general relativity for?

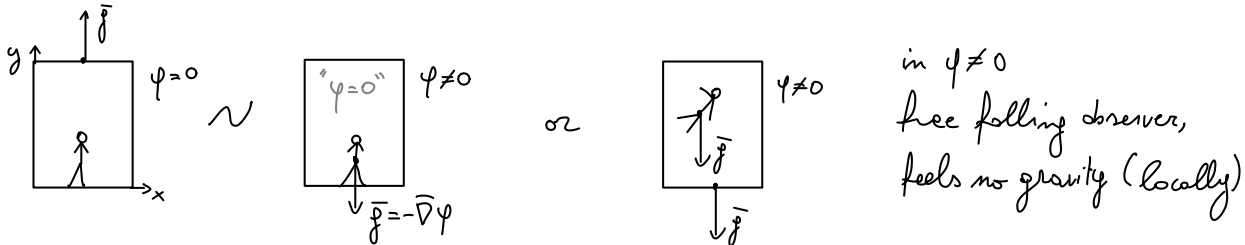
- generalization of special relativity to non-inertial frames
- it describes gravity as a manifestation of the curvature of spacetime

Why do we need it?

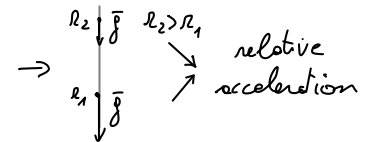
- we need a covariant theory: Lorentz invariance, retardation, ...
- Newton's gravity fails: perihelion shift, gravitational lensing, Grav. waves, time dilation, Lens-Thirring effect, ...
- Newton's gravity is "weird"... $m_I \vec{a} = q \vec{E}$ $m_I \neq q$ chrg in an electric field
 $m_I \vec{a} = -m_G \nabla \varphi$ $m_I = m_G!$ $\Rightarrow m_G$ is associated to the same inertial phenomenon
 inertial mass = gravitational "charge"

The key idea: equivalence principle

- gravity $\longleftrightarrow$ non-inertial frame (Einstein elevator)
- in local inertial frame (free fall) same tensorial expressions of special relativity (SEP)



- real "grav. fields" can not be eliminated everywhere simultaneously by choosing a frame
- you can remove it only locally (the elevator)
- careful here: analogy, not reality. in a real "grav. field" you have tidal forces!



Link between an accelerated frame and the metric

Inertial frame : $ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$

In an accelerated system :

$$\begin{cases} x = x' \cos(\Omega t) - y' \sin(\Omega t) \\ y = x' \sin(\Omega t) + y' \cos(\Omega t) \end{cases}, z = z'$$

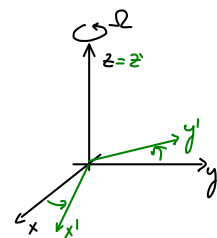
$$\begin{cases} dx = dx' \cos(\Omega t) - x' \sin(\Omega t) \Omega dt - dy' \sin(\Omega t) + y' \cos(\Omega t) \Omega dt \\ dy = dy' \sin(\Omega t) + x' \cos(\Omega t) \Omega dt + dy' \cos(\Omega t) + y' \sin(\Omega t) \Omega dt \end{cases}$$

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

$$= [c^2 - \Omega^2 (x'^2 + y'^2)] dt^2 - dx'^2 - dy'^2 - dz'^2 + 2\Omega y' dx' dt - 2\Omega x' dy' dt$$

$$= g_{\mu\nu} dx^\mu dx^\nu$$

$\uparrow$ time $\leftarrow$ space $\leftarrow$ off-diagonal (mixed)
 g_{00} g_{ii} $g_{0i} g_{i0} g_{ij}$

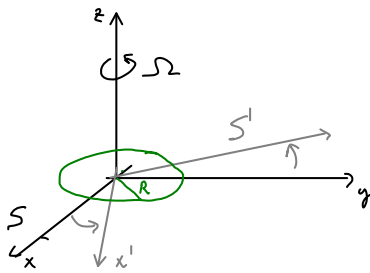


equivalence principle

- Centrifugal force [equivalence principle] $\rightarrow$ locally as a gravitational field
- Mathematically: associate gravity to the metric of the space-time!
- Careful! The metric is still Minkowski, just with a 'wierd coordinate system'

Careful! Here the space is still Minkowski! (flat)

- $g = \eta$ a $\neq$ coord system does not change the properties of space-time
- In fact non-locally, very different behavior than with a grav. field
 - ↳ in a grav. field: for $r \rightarrow \infty$ frame source of field $\bar{g} = -\bar{\nabla}\psi \rightarrow 0$
 - ↳ here $r \rightarrow \infty$ $\bar{g} \rightarrow \infty$!



Note:

measure the circumference of a circle in the x - y plane

$S = \text{inertial}$ ($\Omega = 0$) $C = 2\pi R$

$S = \text{non-inertial}$ ($\Omega \neq 0$) $C \neq 2\pi R$ like in a $\neq$ geometry!
 ↑
 "curved space"

because of Lorentz contraction

Connecting the gravitational field to a property of the space-time

□ Let's try to find the metric producing a Newtonian gravitational field

take a non relativistic particle in a fixed given gravitational field

(1) Newtonian gravity

$$L = -mc^2 + \frac{1}{2}m\vec{v}^2 - m\phi \quad \text{note } m_i \neq m_g$$

$$S_1 = \int L dt = -mc^2 \int \left(1 - \frac{\vec{v}^2}{2c^2} + \frac{\phi}{c^2}\right) dt$$

(2) Free particle with arbitrary metric $g(x^\mu)$

$$S_2 = -mc^2 \int d\tau = -mc^2 \int \frac{1}{c} \left(-g_{\mu\nu} dx^\mu dx^\nu\right)^{1/2}$$

↑
Equivalence principle

NON-free particle in flat space-time ($\eta_{\mu\nu}$) $\sim$ free particle in NON-flat space-time ($g_{\mu\nu}$)

↓
find the metric $g_{\mu\nu}$:

$$\frac{1}{c} \left(-g_{\mu\nu} dx^\mu dx^\nu\right)^{1/2} \stackrel{!}{=} \left(1 - \frac{\vec{v}^2}{2c^2} + \frac{\phi}{c^2}\right) dt \stackrel{*}{\approx} \left(1 - \frac{\vec{v}^2}{2c^2} + \frac{2\phi}{c^2}\right)^{1/2} dt \quad * -\frac{\vec{v}^2}{2c^2} + \frac{\phi}{c^2} \ll 1 \quad \text{i.e. non relativistic particle in weak field}$$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \stackrel{!}{=} -c^2 \left(1 - \frac{\vec{v}^2}{2c^2} + \frac{2\phi}{c^2}\right) dt^2 = -\left(1 + \frac{2\phi}{c^2}\right) c^2 dt^2 + \frac{dx^2}{dt^2} dt^2$$

$$\Rightarrow \boxed{g_{00} = -\left(1 + \frac{2\phi}{c^2}\right), \quad g_{0i} = 0 = g_{i0}, \quad g_{ij} = \delta_{ij}}$$

small correction of $\eta_{00} = -1$

(In reality you have a correction of the same order of g_{00})
 $g_{ii} = 1 - \frac{2\phi}{c^2}$ (coming from Einstein eq. in weak field limit)
 without it you get gravitational lensing wrong!

← identify $g_{\mu\nu}$

□ Reformulate Poisson equation in terms of g and T

$$\overset{(1)}{\nabla^2} \phi = 4\pi G \rho$$

Poisson eq.

$$\overset{(1)}{\nabla^2} g_{00} = -\frac{2}{c^2} \nabla^2 \phi$$

$$\overset{(2)}{T_{00}} = -\rho c^2 \quad (\text{dust})$$

$$\rightarrow \nabla^2 g_{00} = \frac{8\pi G}{c^4} T_{00} \quad \text{double derivatives of } g!$$

$$\square g_{00} = \frac{8\pi G}{c^4} T^0_0 \quad \text{or}$$

assuming: $\frac{1}{c^2} \frac{\delta g_{00}}{\delta t^2} \ll \nabla^2 x$ slow evolution
 $P \ll \rho c^2$ negligible pressure

$$\text{Note: } \square \equiv \sum_{\mu} \partial_{\mu}^2 = -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Provides retardation
negligible in this approximation

$$\nabla^2 \equiv \delta_i^j \partial_j^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

has no retardation

- Careful... this is not yet GR! This is just to show you a link between potential and metric

Non-inertial frames and the geodesic equation

Generic coord. transformation between frames S, S'

$$x'^{\mu} = x'^{\mu}(x^{\mu}(\lambda))$$

↑ transformations here
↑ transformed component

$$\frac{dx'^{\mu}}{d\lambda} = \frac{\partial x'^{\mu}}{\partial x^{\mu}} \frac{dx^{\mu}}{d\lambda} \quad J^{\mu}_{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\mu}} = \text{Jacobian of the transformation}$$

λ : affine parameter $x^{\mu}(\lambda)$

$$\frac{d^2 x'^{\mu}}{d\lambda^2} = \frac{d}{d\lambda} \left(\frac{dx'^{\mu}}{d\lambda} \right) = \frac{d}{d\lambda} \left(\frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{dx^{\alpha}}{d\lambda} \right) = \left(\frac{d}{d\lambda} \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \right) \frac{dx^{\alpha}}{d\lambda} + \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \left(\frac{d^2 x^{\alpha}}{d\lambda^2} \right) = \underbrace{\frac{\partial^2 x'^{\mu}}{\partial x^{\alpha} \partial x^{\beta}} \frac{dx^{\alpha}}{d\lambda} \frac{dx^{\beta}}{d\lambda}}_{(a)} + \underbrace{\frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{d^2 x^{\nu}}{d\lambda^2}}_{(b)} \quad (\text{eq. 1})$$

Inertial frames: special relativity (recall)

• free particle in inertial frame: (c) $\stackrel{!}{=} 0$, (b) $\stackrel{!}{=} 0$, (a) $\stackrel{!}{=} \text{const.}$ $\rightarrow \boxed{\frac{\partial^2 x'^{\mu}}{\partial x^{\alpha} \partial x^{\beta}} \stackrel{!}{=} 0} \rightarrow x'^{\mu} = J^{\mu}_{\mu} x^{\mu} + a'^{\mu}$ i.e. linear transf

Non-inertial frames: general relativity

• free particle: non-inertial frames $\Rightarrow$ (a), (b) arbitrary; (c) $= \frac{d^2 x'^{\mu}}{d\lambda^2} \stackrel{!}{=} 0$ $\left\{ \begin{array}{l} \text{(free-falling frame)} \\ \text{local inertial frame} \\ \text{Equivalence principle} \end{array} \right.$

• eq. (1): $\frac{\partial^2 x'^{\mu}}{\partial x^{\alpha} \partial x^{\beta}} \frac{dx^{\alpha}}{d\lambda} \frac{dx^{\beta}}{d\lambda} + \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{d^2 x^{\nu}}{d\lambda^2} = 0$ $\xrightarrow{\frac{\partial x^{\nu}}{\partial x^{\mu}}}$ $\frac{\partial x^{\nu}}{\partial x^{\mu}} \frac{\partial^2 x'^{\mu}}{\partial x^{\alpha} \partial x^{\beta}} \frac{dx^{\alpha}}{d\lambda} \frac{dx^{\beta}}{d\lambda} + \frac{\partial x^{\nu}}{\partial x^{\mu}} \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{d^2 x^{\nu}}{d\lambda^2} = 0$ $\xrightarrow{1}$ $\Gamma^{\nu}_{\mu\beta} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\beta}}{d\lambda} + \frac{d^2 x^{\nu}}{d\lambda^2} = 0$ Γ affine connection

$$\boxed{\frac{d^2 x^{\nu}}{d\lambda^2} + \Gamma^{\nu}_{\mu\beta} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\beta}}{d\lambda} = 0} \quad \text{geodesic equation: eq. of motion of free particle in a grav. field}$$

- Note:
- $\Gamma^{\nu}_{\mu\beta} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\beta}}{d\lambda}$ plays the role of a force (gravity!) eg. $\lambda = \tau$ $\frac{d^2 x^{\nu}}{d\tau^2} + \Gamma^{\nu}_{\mu\beta} u^{\mu} u^{\beta} = 0$ $\frac{d^2 x^{\nu}}{d\lambda^2}$ 4 acceleration
 - $\frac{d^2 x'^{\mu}}{d\lambda^2} \stackrel{!}{=} 0$ in S' (free falling frame) all laws based on Special Relat hold!
 - $\Gamma^{\nu}_{\mu\beta} = \Gamma^{\nu}_{\beta\mu}$ symmetric with respect to lower indices ($\partial_{\mu}, \partial_{\nu}$ commute) because of $\frac{\partial^2 x^{\nu}}{\partial x^{\mu} \partial x^{\beta}}$ (G.R.: torsion free condition)
 - Transformation that smoothly change from one point to another such to reproduce gravity!! (differential geometry)

Free particle, least action principle

Gravity : $m\vec{g} = m\nabla\phi$ inertial phenomena $\Rightarrow$ free particles = no force!

Free particle : $S = \int L dt = -mc^2 \int d\tau = -mc^2 \int \frac{1}{c} \sqrt{-ds^2} = -mc \int (-\eta_{\mu\nu} dx^\mu dx^\nu)^{1/2}$ $\partial S = 0$: straight trajectory

$\eta_{\mu\nu} \rightarrow g_{\mu\nu}$: $S = - \int (-g_{\mu\nu} dx^\mu dx^\nu)^{1/2}$

$$\partial S = 0 \Rightarrow \boxed{\frac{d^2 x^\nu}{d\lambda^2} + \Gamma_{\mu\beta}^\nu \frac{dx^\mu}{d\lambda} \frac{dx^\beta}{d\lambda} = 0} \quad \text{with} \quad \Gamma_{\mu\beta}^\nu = \frac{1}{2} g^{\nu\gamma} (\delta_\mu \delta_{\gamma\beta} + \delta_\beta \delta_{\gamma\mu} - \delta_\gamma \delta_{\beta\mu})$$

$\Gamma(g)$ = Christoffel symbols

G.R. : Free particles moving in a curved space-time, $g_{\mu\nu}(x^\mu)$ tensorial field, not a scalar one $\phi(x)$
 It works for photons too: gravitational lensing!
 It "inherits" refraction

 Γ is the key! Yes and no...

Yes: it "embodies" the equivalence principle

No: it is not a tensor (it does not transform as such)

: you can choose a frame in which $\Gamma = 0$ locally (free fall).

$$\underbrace{g_{\mu\nu}(P) = \eta_{\mu\nu} + O(|\vec{x} - \vec{x}'|^2)}_{\text{Free falling frame}} \quad \text{i.e.} \quad \underbrace{\delta_{\alpha\beta}(P) = \eta_{\alpha\beta} \quad \delta_{\alpha\beta,\gamma}(P) = 0}_{\text{Locally flat at 1st order}} \quad \Rightarrow \quad \text{need: } \underbrace{\delta_{\alpha\beta,\gamma}(P) \neq 0}_{\text{Typical effects: second derivatives}}$$

Never the less, gravity is there! $\Rightarrow$ We need a quantity based on double derivatives of the $g_{\mu\nu}$

A full theory of gravity

- A covariant theory $\Rightarrow$
 - 1) field eq. based on tensors : $\bar{\nabla}\phi = 4\pi G\rho \rightarrow G_{\mu\nu} = \alpha T_{\mu\nu} \quad \alpha \in \mathbb{R}$ (A non linear theory)
 - 2) $G_{\mu\nu} \stackrel{!}{=} G_{\nu\mu}$ symmetric rank-2 because $T_{\mu\nu} = T_{\nu\mu}$
 - 3) respects energy momentum conservation $\nabla_\nu T^{\mu\nu} = 0 \rightarrow \nabla_\nu G^{\mu\nu} \stackrel{!}{=} 0$
 - 4) $G_{\mu\nu}$ must contain up to second derivatives of g : $g, \delta_\mu g(\Gamma), \delta_\mu \delta_\nu g(\delta_\mu \Gamma)$
 - 5) $G_{\mu\nu} = \alpha T_{\mu\nu}$ must reproduce Newtonian gravity in the weak field limit

We need more sophisticated mathematical tools: differential geometry

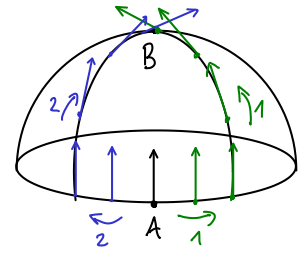
Differential geometry, the language of GR

Curved Space-time: manifold $M = \{P\}$ relative acceleration $\sim$ force

Tangent space $M_P T$ vector space at point $P \in M$

Tensor $T_P: \underbrace{M_P T^* \times \dots \times M_P T^*}_r \times \underbrace{M_P T \times \dots \times M_P T}_s \rightarrow \mathbb{R}$ type $\binom{r}{s}$, rank $N = r + s$ N -linear map

Metric: $g_P: T_P M \times T_P M \rightarrow \mathbb{R}$
depends on position!
 scalar product: $g_P(\bar{v}, \bar{u}) \equiv \langle \bar{v}, \bar{u} \rangle = \underbrace{g_{\mu\nu} v^\mu u^\nu}_{\text{dual vector } \in M_P T^*} = v_\mu u^\mu = v^\mu u_\mu = u_\mu v^\mu$
 norm squared: $g_P(\bar{v}, \bar{v}) = |\bar{v}|^2$
 non degenerate: $g(\bar{u}, \bar{v}) = 0$ with $\bar{v} \neq 0 \Leftrightarrow \bar{u} = 0$



4-interval: $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

Vector: $\bar{v}: M_P T \rightarrow \mathbb{R}$ $\bar{v}(\bar{u}) = \alpha$ $\bar{v} = v^\mu \bar{e}_\mu$ $\{\bar{e}_\mu\}$ basis set of vectors $\bar{e}_\mu \leftarrow \begin{matrix} \text{component} \\ \text{label} \end{matrix}$

Dual vector: $\tilde{v}: M_P T \rightarrow \mathbb{R}$ $\tilde{v}(\bar{v}) = \alpha$ $\tilde{v} = v_\mu \tilde{w}^\mu$ $\{\tilde{w}^\mu\}$ basis set of dual vectors $\tilde{w}^\mu \leftarrow \begin{matrix} \text{label} \\ \text{component} \end{matrix}$

Covariant derivative

$$\begin{aligned} \delta_\mu \bar{v} &= \delta_\mu (v^\alpha \bar{e}_\alpha) = \delta_\mu v^\alpha \bar{e}_\alpha + v^\alpha \delta_\mu \bar{e}_\alpha \quad \leftarrow \text{metric compatibility } \nabla_\beta g_{\mu\nu} \stackrel{!}{=} 0 \\ &= \delta_\mu v^\alpha \bar{e}_\alpha + v^\alpha T_{\mu\alpha}^\gamma \bar{e}_\gamma \quad \alpha \leftrightarrow \gamma \\ &= (\delta_\mu v^\alpha + v^\gamma T_{\mu\gamma}^\alpha) \bar{e}_\alpha = \nabla_\mu v^\alpha \bar{e}_\alpha \end{aligned}$$

is now position dependent
assume $\delta_\mu \bar{e}_\alpha$ to be a vector
 $\delta_\mu \bar{e}_\alpha = T_{\mu\alpha}^\gamma \bar{e}_\gamma$
offline connection coefficients express the change of basis

Make T unique

$$\begin{aligned} 1) T_{\beta\gamma}^\alpha &\equiv T_{\beta\gamma}^\alpha - T_{\gamma\beta}^\alpha \stackrel{!}{=} 0 \quad \text{torsion free} \\ 2) \nabla_\beta g_{\mu\nu} &\stackrel{!}{=} 0 \quad \text{Metric compatibility} \end{aligned} \quad \left. \vphantom{\begin{aligned} 1) T_{\beta\gamma}^\alpha &\equiv T_{\beta\gamma}^\alpha - T_{\gamma\beta}^\alpha \stackrel{!}{=} 0 \\ 2) \nabla_\beta g_{\mu\nu} &\stackrel{!}{=} 0 \end{aligned}} \right\} \text{we link } T \text{ to the metric! } T \Leftrightarrow g \text{ G.R.}$$

$$\begin{aligned} \nabla_b g_{ca} &= \delta_b g_{ca} - T_{bc}^\gamma g_{\gamma a} - T_{ba}^\gamma g_{\gamma c} \stackrel{!}{=} 0 \\ \nabla_c g_{ab} &= \delta_c g_{ab} - T_{ca}^\gamma g_{\gamma b} - T_{cb}^\gamma g_{\gamma a} \stackrel{!}{=} 0 \\ \nabla_a g_{bc} &= \delta_a g_{bc} - T_{ab}^\gamma g_{\gamma c} - T_{ac}^\gamma g_{\gamma b} \stackrel{!}{=} 0 \end{aligned} \quad \begin{aligned} &\text{from metric compatibility} \\ &\downarrow \\ &\text{by cycling the indices} \end{aligned}$$

$$\begin{aligned} T_{bc}^\gamma &= T_{cb}^\gamma \\ g_{ab} &= g_{ba} \end{aligned}$$

$$\delta_b g_{ca} + \delta_c g_{ba} - \delta_a g_{cb} = \underbrace{T_{bc}^\gamma g_{\gamma a}}_+ + \underbrace{T_{ba}^\gamma g_{\gamma c}}_+ + \underbrace{T_{ca}^\gamma g_{\gamma b}}_+ + \underbrace{T_{cb}^\gamma g_{\gamma a}}_+ - \underbrace{T_{ab}^\gamma g_{\gamma c}}_+ - \underbrace{T_{ac}^\gamma g_{\gamma b}}_+ = 2 T_{bc}^\gamma g_{\gamma a}$$

$$\Rightarrow T_{bc}^\gamma = \frac{1}{2} g^{\gamma\delta} (\delta_b g_{\delta c} + \delta_c g_{\delta b} - \delta_\delta g_{bc})$$

Christoffel symbol (of the 2nd kind)
 Christoffel / Levi-civita / Riemannian connection

$$\begin{aligned} \gamma \leftrightarrow \alpha \\ g^{\gamma\delta} \\ g_{\delta\alpha} g^{\delta\gamma} &= \delta_\alpha^\gamma \end{aligned}$$

Directional derivative $\frac{d\bar{v}}{d\lambda} \equiv u^\nu \nabla_\nu \bar{v} \equiv \nabla_{\bar{u}} \bar{v}$

directional derivative (change along direction given by $\bar{u}$)

Parallel transport $u^\nu \nabla_\nu v^\mu = 0$

"shift" $\bar{v}$ along direction $\bar{u}$ "without changing" $\bar{v}$

Geodesic eq $u^\nu \nabla_\nu u^\mu = u^\nu \delta_\nu^\mu + \Gamma_{\nu\gamma}^\mu u^\gamma u^\mu = 0$

generalization of a straight line on curved space
trajectories of free particles in curved space-time
same as for accelerated frames!
equivalence principle, i.e. gravity!

$\hookrightarrow u^\mu = \frac{dx^\mu}{d\lambda}, \frac{dx^\mu}{d\lambda} \delta_\nu^\mu = \frac{d}{d\lambda} \rightarrow$

Riemann "map" $\hat{R}: M_p T \times M_p T \times M_p T \rightarrow M_p T$

$\bar{R}(\bar{v}, \bar{u}, \bar{w}) \equiv \nabla_{\bar{v}} \nabla_{\bar{u}} \bar{w} - \nabla_{\bar{u}} \nabla_{\bar{v}} \bar{w} - \nabla_{[\bar{u}, \bar{v}]} \bar{w}$

Riemann curvature $R: M_p T^* \times M_p T \times M_p T \times M_p T \rightarrow \mathbb{R}$ $R(\tilde{x}, \bar{v}, \bar{u}, \bar{w}) \equiv \hat{x} \hat{R}(\bar{v}, \bar{u}, \bar{w})$

$R^i{}_{jke} = \delta_k \overset{\downarrow}{T}_{ej}^i - \delta_e \overset{\downarrow}{T}_{kj}^i + \overset{\downarrow}{T}_{k\gamma}^i \overset{\downarrow}{T}_{ej}^\gamma - \overset{\downarrow}{T}_{e\gamma}^i \overset{\downarrow}{T}_{kj}^\gamma$ $i, j, k, e = 0, 1, 2, 3$

$= \frac{1}{2} g^{ik} (\delta_k \delta_j g_{er} - \delta_k \delta_r g_{je} - \delta_e \delta_j g_{kr} + \delta_e \delta_r g_{jk})$ $\leftarrow$ double derivatives of g !!

$R^i{}_{jke} = R_{\underbrace{kej}}^i$, $R_{ijke} = -R_{jike} = -R_{ijek}$, $R_{ijke} + R_{ikej} + R_{iej k} = 0$, $R_{[ijke]} = 0$

Ricci tensor $R_{\mu\nu} \equiv g^\alpha_\gamma R^\gamma_{\beta\alpha\nu} = R^\alpha_{\mu\alpha\nu}$

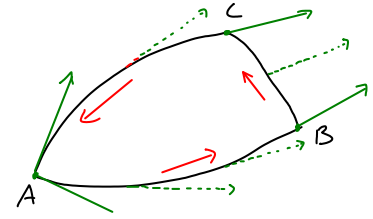
$R_{je} = R_{ej}$ change of volumes along geodesics

Ricci scalar $R \equiv g^{\nu\gamma} R_{\gamma\nu} = R^\nu_\nu$

Trace of the Ricci tensor, "curvature" $[R] = \frac{1}{m^2}$

What does the Riemann tensor describe?

- Parallel transport of a vector along a closed loop
- Difference between initial and final vector
- Explicit Stokes-theorem



1) Parallel transport along $\vec{\partial}_\alpha$ direction:

$$\nabla_\alpha u^\nu = 0 \quad \partial_\alpha u^\nu + T_{\alpha\gamma}^\nu u^\gamma = 0 \quad \partial_\alpha u^\nu = -T_{\alpha\gamma}^\nu u^\gamma \quad (1)$$

2) Change after infinitesimal shift dx^α :

$$du^\nu = \partial_\alpha u^\nu dx^\alpha = -T_{\alpha\gamma}^\nu u^\gamma dx^\alpha$$

3) Total change after a closed loop:

$$\begin{aligned} \Delta u^\nu &\equiv \oint du^\nu && \text{integral over a closed loop} \\ &= - \oint T_{\alpha\gamma}^\nu u^\gamma dx^\alpha && \downarrow B_{\alpha\gamma}^\nu \equiv T_{\gamma\alpha}^\nu \quad (2) \\ &= - \oint B_{\alpha\gamma}^\nu dx^\alpha && \downarrow \text{Stokes theorem! (surface) } dS^{\alpha\beta} \\ &= - \int B_{\alpha\gamma}^\nu dS^{\alpha\beta} && \downarrow \\ &= - \frac{1}{2} \int (\partial_\alpha B_{\beta\gamma}^\nu - \partial_\beta B_{\alpha\gamma}^\nu) dS^{\alpha\beta} && \downarrow \\ &= - \frac{1}{2} \int [\partial_\alpha (T_{\beta\gamma}^\nu u^\gamma) - \partial_\beta (T_{\alpha\gamma}^\nu u^\gamma)] dS^{\alpha\beta} && \downarrow \\ &= - \frac{1}{2} \int [\overset{(2)}{\partial_{\beta\gamma,\alpha}} T_{\alpha\gamma}^\nu u^\gamma + \overset{(1)}{\partial_{\beta\gamma}} T_{\alpha\gamma}^\nu u^\gamma - \overset{(1)}{\partial_{\alpha\gamma,\beta}} T_{\alpha\gamma}^\nu u^\gamma - \overset{(1)}{\partial_{\alpha\gamma}} T_{\beta\gamma}^\nu u^\gamma] dS^{\alpha\beta} \\ &= - \frac{1}{2} \int [\overset{(2)}{\partial_{\beta\gamma,\alpha}} T_{\alpha\gamma}^\nu u^\gamma - \overset{(1)}{\partial_{\beta\gamma}} T_{\alpha\gamma}^\nu u^\gamma - \overset{(1)}{\partial_{\alpha\gamma,\beta}} T_{\alpha\gamma}^\nu u^\gamma + \overset{(1)}{\partial_{\alpha\gamma}} T_{\beta\gamma}^\nu u^\gamma] dS^{\alpha\beta} \\ &= - \frac{1}{2} \int (\underbrace{\partial_{\beta\gamma,\alpha} T_{\alpha\gamma}^\nu - \partial_{\beta\sigma} T_{\alpha\gamma}^\sigma - \partial_{\alpha\gamma,\beta} T_{\alpha\gamma}^\nu + \partial_{\alpha\sigma} T_{\beta\gamma}^\sigma}_{-R_{\gamma\alpha\beta}^\nu} u^\gamma) dS^{\alpha\beta} \end{aligned}$$

$$du^\nu = \frac{1}{2} R_{\gamma\alpha\beta}^\nu u^\gamma dS^{\alpha\beta} \quad *$$

$$R_{\gamma\alpha\beta}^\nu = \partial_{\beta\gamma,\alpha} T_{\alpha\gamma}^\nu - \partial_{\beta\sigma} T_{\alpha\gamma}^\sigma - \partial_{\alpha\gamma,\beta} T_{\alpha\gamma}^\nu + \partial_{\alpha\sigma} T_{\beta\gamma}^\sigma$$

• Note: The change depends on the path

• Physical meaning of $*$: $4\text{-acceleration} \hat{=} \frac{d^2 x^\nu}{d\tau^2} = \overset{S^\nu \text{ is tidal stress tensor (tidal forces)}}{R_{\gamma\alpha\beta}^\nu} u^\gamma u^\alpha dl^\beta$ geodesic deviation eq.

A theory of gravity build on that: GR

Recall:

- 1) field eq. based on tensors: $\bar{\nabla}\phi = 4\pi G\rho \rightarrow G_{\mu\nu} = \alpha T_{\mu\nu} \quad \alpha \in \mathbb{R}$ (A non linear theory)
- 2) $G_{\mu\nu} \stackrel{!}{=} G_{\nu\mu}$ symmetric rank-2 because $T_{\mu\nu} = T_{\nu\mu}$
- 3) respects energy momentum conservation $\nabla_\nu T^{\mu\nu} = 0 \rightarrow \nabla_\nu G^{\mu\nu} \stackrel{!}{=} 0$
- 4) $G_{\mu\nu}$ must contain up to second derivatives of g : $g, \delta_\mu g (\nabla), \delta_\mu \delta_\nu g (\delta, \nabla)$
- 5) $G_{\mu\nu} = \alpha T_{\mu\nu}$ must reproduce Newtonian gravity in the weak field limit

* The only tensor satisfying these conditions is $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ Einstein tensor

Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \quad \leftarrow \text{free to add } \Lambda g_{\mu\nu} \text{ Cosmological const.}$$

equivalent ∇

$$R_{\mu\nu} = \frac{8\pi G}{c^4} (T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}) \quad T \equiv T^\mu{}_\mu \quad * \text{ see appendix}$$

Lovebuck theorem

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{4\pi G}{c^4} T_{\mu\nu} \quad \text{unique 4D rank 2 tensor field theory satisfying the above conditions}$$

Einstein-Hilbert action

$$S_H = \int R \sqrt{-g} d\Omega \quad \delta(S_H + S_m) = 0 \rightarrow \text{Einstein field equations (eq of motion of the field: } g_{\mu\nu})$$

Energy-Momentum conservation

$$\nabla_\nu T^{\mu\nu} = \delta_\nu T^{\mu\nu} + \underbrace{T^{\mu\nu}}_{\text{non conservation}} \nabla_\nu = 0 \quad \neq \delta_\nu T^{\mu\nu}!$$

Strong equivalence principle: $\eta \rightarrow g, \delta_\nu \rightarrow \nabla_\nu$ (e.g. electrodynamics in curved space-time)

— o —

Conceptually all boils down to:

$$\nabla^2 \phi = 4\pi G \rho$$

$$\square \equiv \delta_\nu \delta^\nu = -\frac{1}{c^2} \frac{\delta^2}{\delta t^2} + \frac{\delta^2}{\delta x^1} + \frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta x^3} \quad \text{Retardation}$$

$$\delta_\nu \delta^\nu \phi = \frac{4\pi G}{c^2} T_\nu{}^\nu$$

$$\rho \simeq \frac{T_\nu{}^\nu}{c^2} \quad \text{non relativistic pressureless ideal fluid } (\rho_0)$$

$$G_{\mu\nu} = \frac{4\pi G}{c^4} T_{\mu\nu}$$

$$\eta_{\mu\nu} \rightarrow g_{\mu\nu} \quad \delta_\mu \rightarrow \nabla_\mu \quad \text{Strong Equivalence Principle (SEP)}$$

$$S_{SR} = \int \sqrt{-\eta_{\mu\nu}} dx^\mu dx^\nu \rightarrow S_{GR} = \int \sqrt{-g_{\mu\nu}} dx^\mu dx^\nu$$

Conceptual steps and summary

The equivalence principle $m_G = m_I$

accelerations due to gravity $\sim$ non inertial frame $S_G = S_{free} \Rightarrow g_{\mu\nu} \neq \eta_{\mu\nu}$



Generic coordinate transformation $x^{\mu'} = x^{\mu'}(x^{\mu}(\lambda))$

$$\frac{d^2 x^{\mu'}}{d\lambda^2} = \frac{\partial^2 x^{\mu'}}{\partial x^{\alpha} \partial x^{\beta}} \frac{dx^{\alpha}}{d\lambda} \frac{dx^{\beta}}{d\lambda} + \frac{\partial x^{\mu'}}{\partial x^{\nu}} \frac{d^2 x^{\nu}}{d\lambda^2} \Rightarrow \left[\begin{array}{l} \text{local inertial frame } (x^{\mu}): \frac{d^2 x^{\mu'}}{d\lambda^2} = 0 \\ \text{arbitrary frame } (x^{\mu}) \end{array} \right] \quad \boxed{\frac{d^2 x^{\nu}}{d\lambda^2} + \Gamma_{\beta\mu}^{\nu} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\beta}}{d\lambda} = 0}$$



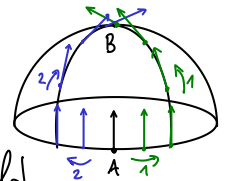
Free particle

$$S = -mc \int (-g_{\mu\nu} dx^{\mu} dx^{\nu})^{1/2}, \quad \delta S = 0 \Rightarrow \boxed{\frac{d^2 x^{\nu}}{d\lambda^2} + \Gamma_{\beta\mu}^{\nu} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\beta}}{d\lambda} = 0} \quad \Gamma_{\alpha\beta}^{\gamma} = \frac{1}{2} g^{\gamma\mu} (\partial_{\alpha} g_{\mu\beta} + \partial_{\beta} g_{\mu\alpha} - \partial_{\mu} g_{\alpha\beta}) \text{ Christoffel symbols}$$



Generic transformation $\rightarrow$ Need differential geometry

Properties of the space-time differ point by point, Non Minkowski manifold



Covariant derivative

$$\nabla_{\mu} v^{\alpha} = \partial_{\mu} v^{\alpha} \bar{e}_{\alpha} + v^{\alpha} \partial_{\mu} \bar{e}_{\alpha} = (\partial_{\mu} v^{\alpha} + v^{\gamma} \Gamma_{\mu\gamma}^{\alpha}) \bar{e}_{\alpha} = D_{\mu} v^{\alpha} \bar{e}_{\alpha}$$

Parallel transport and Geodesic eq.

$$u^{\nu} \nabla_{\nu} u^{\mu} = 0 \quad \text{generalization of a straight line}$$

Riemann curvature Tensor

$$\Delta u^{\nu} \equiv \oint \Delta u^{\nu} \quad R_{\gamma\alpha\beta}^{\nu} = \Gamma_{\beta\gamma,\alpha}^{\nu} - \Gamma_{\alpha\gamma,\beta}^{\nu} + \Gamma_{\alpha\sigma}^{\nu} \Gamma_{\beta\gamma}^{\sigma} - \Gamma_{\beta\sigma}^{\nu} \Gamma_{\alpha\gamma}^{\sigma}$$



Torsion free + Metric compatibility

$$\Gamma_{\alpha\beta}^{\gamma} = \frac{1}{2} g^{\gamma\mu} (\partial_{\alpha} g_{\mu\beta} + \partial_{\beta} g_{\mu\alpha} - \partial_{\mu} g_{\alpha\beta}) \quad \text{Christoffel symbols}$$



Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Space-time tells matter how to move, matter tells S-T how to "curve"

Particle in a gravitational field $\rightarrow$ Free particle in a curved space time

Appendix

Einstein eq.s can be written as:

$$g^{\mu\nu} G_{\mu\nu} = g^{\mu\nu} R_{\mu\nu} - \frac{1}{2} R g^{\mu\nu} g_{\mu\nu} = \kappa g^{\mu\nu} T_{\mu\nu} \quad R - \frac{1}{2} 4R = \kappa T \quad \boxed{R = -\kappa T}$$

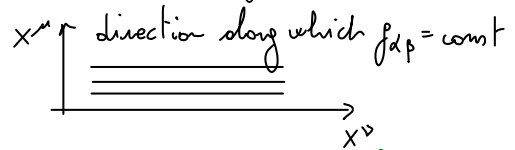
$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = R_{\mu\nu} + \frac{\kappa}{2} T g_{\mu\nu} = T_{\mu\nu} \quad \Rightarrow \quad \boxed{R_{\mu\nu} = \kappa \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right)}$$

- Convenient for vacuum ($T_{\mu\nu} = 0$) solutions where they reduce to $R_{\mu\nu} = 0$
- " in cosmology where T can be easily found

A glimpse in Energy-momentum "conservation" in GR

$$\frac{du_\nu}{d\tau} - \frac{1}{2} \delta_\nu \delta_\alpha \delta_\beta u^\alpha u^\beta = 0 \quad \text{Geodesic eq. Torsion free + Metric compatibility (see previous notes)}$$

$$p^\alpha \equiv m u^\alpha \quad \Rightarrow \quad \boxed{\frac{dp_\nu}{d\tau} = \frac{m}{2} \delta_\nu \delta_\alpha \delta_\beta u^\alpha u^\beta}$$



$\uparrow v=1$ cartesian c.
 $v=\phi$ polar c.

$$\Rightarrow g_{\alpha\beta, \nu} = 0 \Rightarrow P_0 \text{ is conserved! i.e. metric does not depend on } x^\nu$$

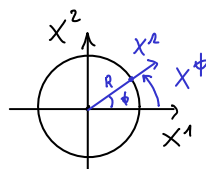
such conservation is due to the symmetries of the system (i.e. in the metric)

- This is just a glimpse, for more look for Killing vector fields, Killing equations, Lie derivatives, Isometries

e.g.: $g_{\alpha\beta, 0} = 0$ (no time dependency, i.e. g is stationary) $P_0 = g_{0\nu} P^\nu = \text{const} = -\frac{E}{c}$ (not $P^0 = \text{const}$) local inertial rest frame
universe expands $\Leftrightarrow g$ evolves with time $\Rightarrow$ energy of photons is not conserved: cosmological redshift

e.g.: $g_{\alpha\beta, \phi} = 0$ (in spherical coordinates) $\Rightarrow P_\phi = g_{\phi\phi} P^\phi = r^2 m \frac{d\phi}{d\tau} = \text{const.}$ angular momentum!

$\delta_\phi g_{\alpha\beta} = 0 \quad g_{\alpha\beta} = \text{const for a fixed } r$
(spherical symmetry)



(x^1, x^2) cartesian coord.
 (x^r, x^ϕ) spherical coord.

Part II

The homogeneous Universe

Relativistic cosmological model

- A model describing the general behaviour of the Universe
- The available interactions

Theory	Interaction	Behavior	Typical scale
QCD	Strong	$1/r^7$	1.4×10^{-15} m
Electroweak	Weak	$1/r^5$ to $1/r^7$	10^{-18} m
QED	Electromagnetic	$1/r^2$	∞
GR	Gravity	$1/r^2$	∞

irrelevant
irrelevant
irrelevant
yes!

{ The universe is neutral
on large scales

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}$$

(2) geometry (1) "matter" source of gravity

- Assumptions: homogeneity + isotropy on large scales = cosmological principle (observations)

(1) The "matter" term $T_{\mu\nu}$

a) Radiation photons

b) Matter { baryons $P \neq 0$ it has pressure (relevant on "small" scales)
dark matter $P = 0$ (dust) only gravitational effect

↳ direct detection : so far not successful

↳ production at LHC : " " " "

↳ evidence from : gravitational interactions:

- velocity dispersion of galaxies in galaxy clusters
- temperature of intra-cluster gas " " "
- rotation curves of individual galaxies
- gravitational lensing of galaxies and galaxy clusters
- cosmic structure formation
- dynamic of the universe

c) Cosmological const. OR Dark energy: needed to explain accelerated expansion of the universe
 Λ DE $\Rightarrow \Lambda$ CDM model or w CDM model

Treat them as ideal fluids, yes also Λ

$$T_{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) u_\mu u_\nu + P g_{\mu\nu}$$

$$P = w \rho c^2$$

parametric eq. of state
eq. of state parameter w

Cosmological constant as a (effective) fluid

We associate an energy-momentum tensor to Λ

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \quad \text{or} \quad R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} (T_{\mu\nu} + T_{\mu\nu}^{\Lambda})$$

interpret Λ as a fluid

$$T_{\mu\nu}^{\Lambda} = \underbrace{-\frac{c^4 \Lambda}{8\pi G}}_P g_{\mu\nu} \stackrel{!}{=} \left(\underbrace{\rho}_{\stackrel{!}{=} 0} + \frac{P}{c^2} \right) u_{\mu} u_{\nu} + P g_{\mu\nu} \Rightarrow$$

$$\stackrel{!}{=} P g_{\mu\nu} \stackrel{!}{=} -\rho c^2 g_{\mu\nu}$$

$$\Rightarrow T_{\mu\nu}^{\Lambda} = -\frac{c^4 \Lambda}{8\pi G} g_{\mu\nu}$$

$$\rho_{\Lambda} = -P_{\Lambda}$$

$$P = w \rho c^2 \Rightarrow$$

$$w_{\Lambda} = -1$$

$$\rho_{\Lambda} = \frac{c^2 \Lambda}{8\pi G}$$

constant density!
negative pressure!

Note on the cosmological principle

Λ automatically satisfy the cosmological principle: no dependency on position or directions
DE must satisfy certain conditions we have to impose

A deeper interpretation of Λ

- GR is an effective theory: it describes macroscopic behaviour of the underlined quantum realm (QFT)
- What is Λ associated to? Λ is a property of space-time: "empty space"
 $T_{\mu\nu}^{\Lambda}$ has no dependency on observer motion } vacuum, ground state
- Effective properties of vacuum:

1) Thermodynamic arguments

A universe with "nothing" $\Rightarrow$ only vacuum with density $\hat{\rho} \neq 0$

$$\left. \begin{array}{l} \text{energy in the Universe } dE = \rho c^2 dV \\ 1^{\text{st}} \text{ law of thermodynamic } dE + p dV = 0 \end{array} \right\} \rho c^2 dV + p dV = 0 : p = -\rho c^2 \quad w = -1!$$

2) Symmetry properties of vacuum

In free falling frame $\rightarrow$ locally Minkowski $\eta_{\mu\nu}$

vacuum is Lorentz invariant in $\eta_{\mu\nu}$ (actually invariant under the Poincaré group which is broader)

$\Rightarrow$ insensitive to such transformations (which depend on $\bar{u}$), i.e. its energy density is always the same

$$\hookrightarrow T_{\mu\nu} \propto \eta_{\mu\nu} \quad (\text{recall: the Lorentz transf. } \Lambda_{\mu}^{\nu} \text{ are those for which } \eta' = \Lambda^T \eta \Lambda = \eta!)$$

$$\boxed{T_{\mu\nu} \propto g_{\mu\nu}} \quad (\text{recall: strong equivalence principle}) \quad (\text{free falling observer experience } \eta)$$

Note:

(2 fully degenerate parameters)

We might have Λ and vacuum but they can not disentangled: $(\Lambda + \alpha) g_{\mu\nu} \Rightarrow \rho_{\text{eff}} = \frac{c^2 \Lambda}{8\pi G} + \rho_{\text{vac}}$

But... if $\rho_\Lambda \sim \rho_{\text{vac}}$, we get a problem....

- (a) from cosmological observations: $|\Lambda| < 10^{-56} \text{ cm}^{-2} \sim m_\Lambda < 10^{-32} \text{ eV}$
 "unnaturally" small but not zero ^{*2} $\rightarrow$ fine tuning problem
 - (b) from a QFT perspective : vacuum energy Ω_v estimate $\rightarrow \Omega_v \sim 10^{121}$ times larger!
 - (c) from cosmological observations: $\Omega_\Lambda \sim \Omega_m$ today: why today?! $\rightarrow$ coincidence problem
- * (eg. photon with lowest possible energy $\lambda = R_H = \frac{c}{H_0} \approx 36 \text{ pc}/h \Rightarrow E_\gamma \sim 10^{-29} \text{ eV}$)

This is why people substitutes Λ with Dark-Energy (DE): a dynamic scalar field
 not a constant, it can evolve with time: $w(z)$ $\leftarrow$

In any case, behaviour of DE must be very close to Λ : so far observations are compatible with Λ !

More on that later See "scalar fields in cosmology"

Energy-density evolution of the components

Apply cosmological principle

$\Rightarrow$ no gradients, no fluxes

$\Rightarrow$ matter remains at rest in comoving coordinates: i.e. fixed $r, \theta, \phi \Rightarrow \bar{u} = (c, 0, 0, 0)^T$

$\Rightarrow T = \text{diag}(\rho c^2, p, p, p) \quad T_{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u_\mu u_\nu + p g_{\mu\nu}$

Assume: components are not interacting: densities evolve independently

$\nabla_\mu (T_\mu^{\mu\nu} + T_m^{\mu\nu} + T_\Lambda^{\mu\nu}) = 0 \quad \nabla_\mu T_i^{\mu\nu} = j_i^{\mu\nu} \stackrel{!}{=} 0$

$\nabla_\mu T^{\mu 0} = \dot{\rho} c^2 + 3H(\rho c^2 + p) = 0 \quad \Leftrightarrow \quad \frac{d(\rho c^2 \bar{a}^3)}{dt} + p \frac{d\bar{a}^3}{dt} = 0$

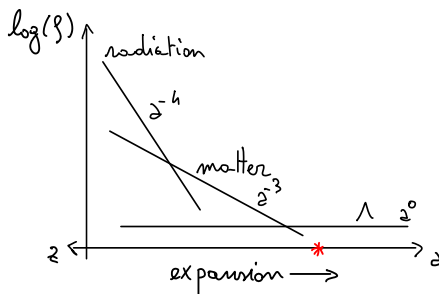
$\bar{a}$: length scale $E = \rho c^2 V$
 $V = V_0 \bar{a}^3$ i.e. $dE + p dV = 0$
 1st law of thermodynamics
 Adiabatic condition

Density evolution, exploit $p_i = w_i \rho_i c^2$

$\frac{d(\rho c^2 \bar{a}^3)}{dt} + p \frac{d\bar{a}^3}{dt} = 0 \quad d(\rho \bar{a}^3) + w \rho \bar{a}^3 d\bar{a}^3 = 0 \quad \rho \bar{a}^3 + 3\rho \bar{a}^3 \frac{d\bar{a}}{\bar{a}} + w \rho \bar{a}^3 \frac{d\bar{a}}{\bar{a}} = 0 \quad \frac{d\rho}{\rho} = -3(1+w) \frac{d\bar{a}}{\bar{a}}$

$\log \rho = -3(1+w) \log \bar{a} + C \quad \rho = \rho_0 \bar{a}^{-3(1+w)}$

$w = 0$	dust	$\rho_m = \rho_{m0} \bar{a}^{-3}$	matter (also baryons on large scales)
$w = 1/3$	relativistic species *	$\rho_r = \rho_{r0} \bar{a}^{-4}$	photons, neutrinos * (γ : Bose-Einstein distrib. η : relativistic Fermi-Dirac)
$w = -1$	cosmo. const.	$\rho_\Lambda = \text{const}$	vacuum
$w(t) = \frac{p(t)}{\rho(t)c^2}$	dark energy	$\rho_{DE} = \text{evolves}$	dynamical fluid



3 eras: radiation domination era
 matter domination era
 Λ domination era (de Sitter universe)

Multi-component cosmological fluid

We can define an effective $T_{\mu\nu}$ containing all species

$T_{\mu\nu} = \sum_i (T_{\mu\nu})_i$ i-th component $i = \text{Baryons, Dark-matter, Radiation, } \Lambda$ (or Dark Energy)

$= \sum_i \left[\left(\rho_i + \frac{p_i}{c^2}\right) u_\mu u_\nu + p_i g_{\mu\nu} \right]$

$= u_\mu u_\nu \left(\sum_i \rho_i + \sum_i \frac{p_i}{c^2} \right) + g_{\mu\nu} \sum_i p_i \Rightarrow \text{behaves as one single fluid with } \rho = \sum_i \rho_i \quad p = \sum_i p_i$

$\Rightarrow R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu}$ Contains all fluids!

(2) The "geometry" term $R_{\mu\nu}$

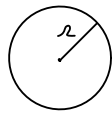
Impose symmetries to $g_{\mu\nu}$ to solve Einstein eq.s analytically: homogeneity + isotropy
 Assume: non static universe

- 1) Fundamental observer (free fall): $ds^2 = -c^2 dt^2 = g_{00}(dx^0)^2 \Rightarrow g_{00} = -1$ sets time units
- 2) Isotropy, no privileged direction: $g_{0i} = 0$ $g_{ij} = 0$ $i \neq j$ clocks synchronization
 most general isotropic metric $\rightarrow h_{ij} dx^i dx^j = f(r) dr^2 + r^2 d\Omega^2$ like Schwarzschild $\Rightarrow$ same $G_{\mu\nu}$ with $A(r)=0$

Spherical coordinates: $\bar{x} = (r, \theta, \phi)^T$

- flat space, observer in the center

$$d\ell^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2(d\theta^2 + \sin^2(\theta) d\phi^2) \quad \text{spatial interval}$$



$$S = 2\pi r \quad \text{circumference}$$

$$A = 4\pi r^2 \quad \text{surface of sphere, } dA = r^2 \sin\theta d\theta d\phi \quad \text{surface element}$$

- Curved geometry, distinct "radiuses": R, r !

$$d\ell^2 = dR^2 + r^2(d\theta^2 + \sin^2(\theta) d\phi^2)$$

radiuses for a fixed R (r in "angular" coordinates)
 θ, ϕ ($dR = f(r) dr$)

Here, a sphere has radius R but circumference $= 2\pi r$ and surface $= 4\pi r^2$

$$\Rightarrow h_{11} = f^2(r) \quad h_{22} = r^2 \quad h_{33} = r^2 \sin^2(\theta)$$

- 3) Dynamic, isotropic expansion: $d\ell^2 = a^2(t)(h_{ij} dx^i dx^j)$ $a(t)$ = scale factor
 rescaled by a time dependent factor only $\Rightarrow$ dynamics!
 set distance units such that $a(\text{today}) \stackrel{!}{=} 1$
 expansion $\Rightarrow a(t) < 1$ for $t < t_{\text{today}}$

4) Homogeneity:

$$* (G_i^i = R_i^i - \frac{1}{2} R g_i^i = -\frac{1}{2} R_i^i)$$

* See appendix
 $g_{\mu\nu} \rightarrow \bar{g}_{\mu\nu} \rightarrow R_{\mu\nu}$

$R_i^i = \text{const} \Leftrightarrow G_i^i = \text{const} \equiv 3\kappa$ on $t = \text{const}$ hypersurfaces ($i=1,2,3$)
 i.e. Same spatial curvature on a time slice: foliation

$$G_i^i = g^{ij} G_{ij} = f^2(r) G_{rr} + r^2 G_{\theta\theta} + r^2 \sin^2\theta G_{\phi\phi} = -\frac{1}{r^2} [r(1-f^2(r))] \stackrel{!}{=} 3\kappa$$

$$\text{integrate: } r(1-f^2(r)) = -\frac{3\kappa}{2} r^2 + \frac{C}{r} \quad f^2(r) = g_{rr} = (1 + \kappa r^2)^{-1}$$

[initial condition: flatness at $r=0 \Rightarrow C=0$]
 "on small scales you see no curvature"

⇒ Friedman-Lemaître-Robertson-Walker metric (FLRW)

$$ds^2 = -c^2 dt^2 + a^2(t) \left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right)$$

r, θ, ϕ = comoving coordinates

• $k \in \mathbb{R}$ ($k < 0, k = 0, k > 0$) but a can always be rescaled such that:

$$\left(k \rightarrow \frac{k}{|k|} \quad a \rightarrow \sqrt{|k|} a \quad c \rightarrow \frac{c}{\sqrt{|k|}} \quad \text{leave the 4 interval the same} \right)$$

eg. $ka^2 = \frac{k}{|k|} \cdot |k| a^2$

names are from geometry of spatial surfaces with $t = \text{const.}$

$$k = \begin{cases} 0 & \text{flat} \\ 1 & \text{closed / spherical} \\ -1 & \text{open / hyperbolic} \end{cases}$$

• $k=1$: define parameter χ : $(d\chi)^2 \equiv \frac{(dr)^2}{1-r^2} \Rightarrow a = \sin \chi$ (spherical geometry)

$$(\chi=0 \rightarrow a=0) \quad (\chi=\frac{\pi}{2} \rightarrow a=1) \quad (\chi=\pi \rightarrow a=0) \Rightarrow a \in [0,1] \text{ closed universe}$$

$$d\tilde{r}^2 = d\chi^2 + \sin^2 \chi d\Omega^2 \quad \text{surface element of 3D sphere embedded in 4D space}$$

$$3D \text{ volume element: } V = \int \sqrt{g_{(3)}} d\chi^3 = \int \sin^2 \chi d\Omega^2 d\chi = 4\pi \int_0^\pi \sin^2 \chi d\chi = 2\pi^2 \quad (\text{finite})$$

• $k=-1$: as above: $(d\chi)^2 \equiv \frac{(dr)^2}{1+r^2} \Rightarrow a = \sinh \chi$ (hyperbolic geometry)

$$(\chi=0 \rightarrow a=0) \quad (\chi=\infty \rightarrow a=\infty) \Rightarrow a \in [0, \infty) \text{ open universe}$$

$$V = 4\pi \sinh^3 \chi \rightarrow \infty \text{ for } \chi \rightarrow \infty \quad (\text{infinite})$$

$$\Rightarrow ds^2 = -c^2 dt^2 + a^2(t) \left[d\chi^2 + f_k^2(\chi) (d\theta^2 + \sin^2 \theta d\phi^2) \right] \quad f_k(\chi) = \begin{cases} \chi & k=0 \\ \sin \chi & k=1 \\ \sinh \chi & k=-1 \end{cases}$$

$$ds^2 = a^2(t) \left[-c^2 d\tau^2 + d\chi^2 + f_k^2(\chi) (d\theta^2 + \sin^2 \theta d\phi^2) \right] \quad d\tau \equiv \frac{dt}{a(t)} \quad \tau = \frac{\text{conformal time}}{\text{not the proper time}}$$

Plug the metric to find the "geometry" terms: (see Appendix)

$$1) T_{11}^0 = \frac{\partial \dot{a}}{(1-kr^2)c}$$

$$T_{0i}^i = T_{i0}^i = \frac{\dot{a}}{c}$$

$$T_{12}^2 = T_{13}^3 = T_{21}^1 = T_{31}^1 = \frac{1}{a}$$

$$T_{23}^3 = T_{32}^3 = \omega \theta$$

$$T_{22}^0 = \frac{\partial \dot{a}}{c} r^2$$

$$T_{11}^1 = \frac{-kr}{(1-kr^2)}$$

$$T_{33}^2 = -\sin \theta \cos \theta$$

$$T_{33}^0 = T_{22}^0 \sin^2 \theta$$

$$T_{22}^1 = -r(1-kr^2)$$

$$T_{33}^1 = T_{22}^1 \sin^2 \theta$$

factors $\frac{1}{c}$ come from $\delta_0 = \frac{\delta}{c \delta t}$ $\dot{a} = \frac{da}{dt}$

$$2) R_{00} = -3 \frac{\ddot{a}}{a^3} \quad R_{11} = \frac{\partial \ddot{a} + 2\dot{a}^2 + 2k}{(1-kr^2)a^3} \quad R_{22} = r^2 \left(\frac{\partial \ddot{a}}{a^3} + 2 \frac{\dot{a}^2}{a^3} + 2k \right) \quad R_{33} = r^2 \left(\frac{\partial \ddot{a}}{a^3} + 2 \frac{\dot{a}^2}{a^3} + 2k \right) \sin^2 \theta$$

$$3) R = \frac{6}{a^3} \left(\frac{\partial \ddot{a}}{a^3} + \frac{\dot{a}^2}{a^3} + k \right) \quad \text{note: } R^i_i = \frac{6k}{a^3(t)^2} \text{ curvature evolves but type } (k=-1, 0, 1) \text{ does not}$$

Final field equations: dynamic of the universe $a(t)$

- Einstein field eq.s (convenient in this form): $R_{\mu\nu} = \kappa \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right)$ $(T_{\mu\nu}) = \text{diag}(\rho c^2, P, P, P)$
 $T^\mu{}_\mu = T = -\rho c^2 + 3P$

The only non trivial solutions are:

(1) $\mu\nu=00$ $-3 \frac{\ddot{a}}{a^3} = \frac{8\pi G}{c^4} \left(\rho c^2 - \frac{\rho c^2}{2} + \frac{3}{2} P \right) = \frac{4\pi G}{c^2} \left(\rho + 3 \frac{P}{c^2} \right)$ $\frac{\ddot{a}}{a^3} = -\frac{4\pi G}{3c^2} \left(\rho + 3 \frac{P}{c^2} \right)$

(2) $\mu\nu=ii$ all the same because of isotropy (eq. $\mu\nu=22$) $\frac{\ddot{a}}{a^3} + 2 \left(\frac{\dot{a}}{a} \right)^2 + \frac{2\kappa}{a^2} = \frac{4\pi G}{c^2} \left(\rho - \frac{P}{c^2} \right)$

(1) $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left(\rho + 3 \frac{P}{c^2} \right)$ Friedmann eq.s.

(2) $H^2 = \frac{8\pi G}{3} \rho - \frac{\kappa}{a^2} = \frac{8\pi G}{3} (\rho_m + \rho_r) - \frac{\kappa}{a^2} + \frac{\Lambda}{3}$ $(H \equiv \frac{\dot{a}}{a} \text{ Hubble function})$
 $\rho = \rho_m + \rho_r + \rho_\Lambda$

- Friedmann eq.s are not independent: using $\frac{d(\rho a^3)}{dt} + p \frac{da^3}{dt} = 0$ you can obtain (1) from (2)
 plug (1) in (2) $\Rightarrow$

Important quantities and condition for flat universe:

- Friedmann (2): Set $\kappa \stackrel{!}{=} 0 \Rightarrow \rho_c \equiv \frac{3H^2}{8\pi G}$ critical density ρ for which the universe is flat

- Friedmann (2): $1 = \frac{8\pi G}{3H^2} \rho - \frac{\kappa c^2}{a^2 H^2} = \Omega + \Omega_\kappa$ with $\Omega \equiv \rho/\rho_c$ density contrast
 $\Omega_\kappa \equiv -\frac{\kappa c^2}{a^2 H^2}$ density contrast of curvature

Condition for a flat universe ($\kappa=0$):

$$\Omega_\kappa = 0 \Leftrightarrow \Omega = \Omega_m + \Omega_r + \Omega_\Lambda = 1$$

- Friedmann (2): $H^2 = \frac{8\pi G}{3c^2} (\rho_m a^{-3} + \rho_r a^{-4} - \kappa a^{-2} + \rho_\Lambda) = H_0^2 (\Omega_m a^{-3} + \Omega_r a^{-4} - \Omega_\kappa a^{-2} + \Omega_\Lambda) \equiv H_0^2 E(a)$
 $a(t_0) \equiv a_0 \stackrel{!}{=} 1$ today

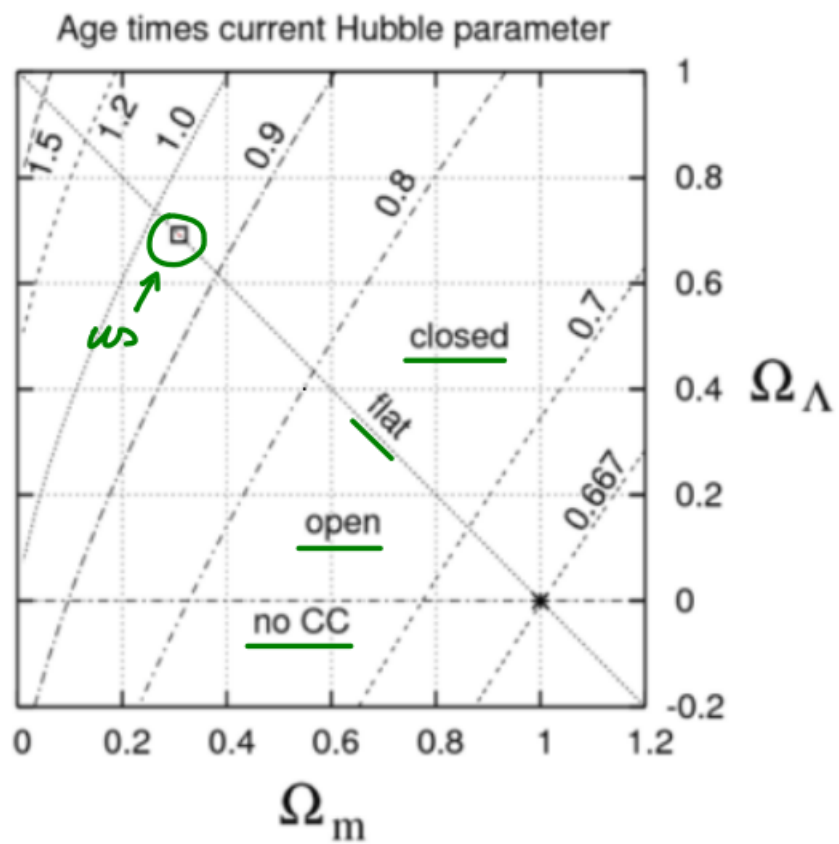
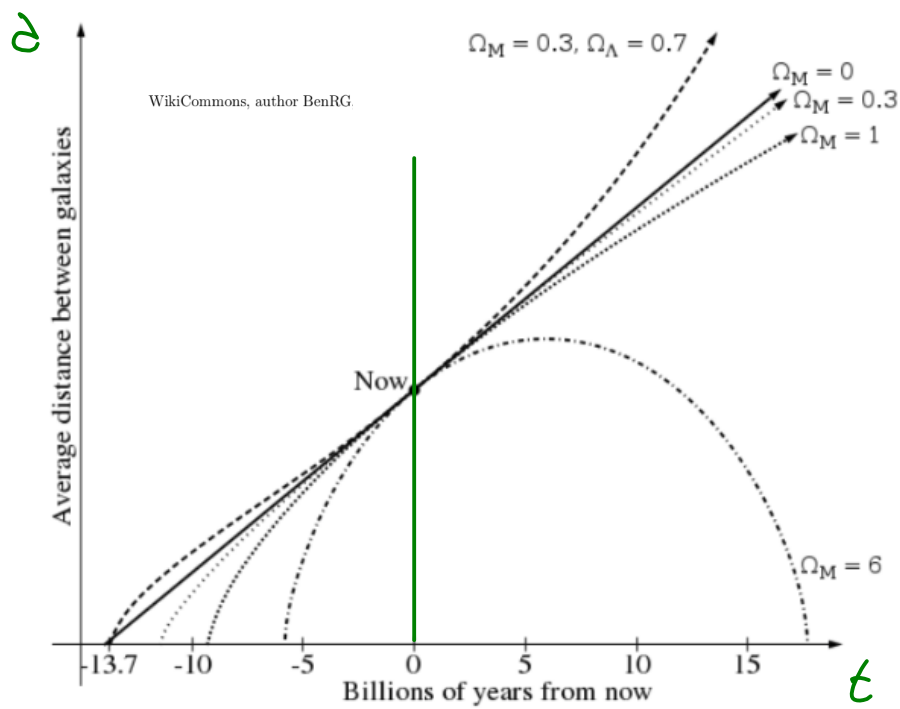
Hubble constant: H_0 Current expansion rate (today, $a=1$) $[H_0] = \text{km s}^{-1} \text{Mpc}^{-1} = \text{s}^{-1}$
 h you also find $H_0 = 100 h \text{ km/s/Mpc}$ with, for example, $h = 0.7$

Hubble law: $D_H = \frac{c}{H_0} z$ at low z linear relationship between D and z

Hubble time: $t_H \equiv H_0^{-1} = 13.9 \text{ Gyr}$

Hubble radius: $r_H = c \cdot H_0^{-1} = 4.2 \text{ Gpc}$

- Measurements at $a=1$ (today): $H_0 \approx 70 \text{ km/s/Mpc}$ $\Omega_m \sim 0.3$ $\Omega_\Lambda \sim 0.7$ $\Omega_\kappa \sim 0$ flat
 $\rho_c \equiv \frac{3H_0^2}{8\pi G} = 1.88 h^2 \times 10^{-27} \text{ g cm}^{-3}$ Earth: $\rho \sim 5 \text{ g cm}^{-3}$ Galaxy: $\rho \sim 10^{-24} \text{ g cm}^{-3}$



Cosmological redshift

• Redshift, z

"o" = observed (use $t_o = \text{today}$, $a_o = 1$)

$$ds^2 = 0 = -c^2 dt^2 + a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right) = -c^2 dt^2 + a^2(t) D \quad D = \text{const (comoving coord.)}$$

$\Rightarrow dt \propto \frac{1}{a}$ the "time note" changes because of the time evolution of the metric (!)

$\Rightarrow$ this affects frequencies ν : $[\nu] = s^{-1}$

$$z \equiv \frac{\lambda_o - \lambda}{\lambda} = \frac{\lambda_o}{\lambda} - 1 = \frac{\nu}{\nu_o} - 1 = \frac{dt_o}{dt} - 1 = \frac{a(t_o)}{a(t)} - 1 \Rightarrow \boxed{a(z) = (1+z)^{-1}} \quad \underline{a = \text{direct observable!}}$$

Photons lose energy as the universe expands $a \uparrow \Rightarrow E = h\nu \downarrow$

Where is the energy of the photons gone?!

• Energy of photons decreases with time because of the time evolution of g !

This is due to the fact that in GR "energy is not conserved"

$$\nabla_\mu T^{\mu 0} = \partial_\mu T^{\mu 0} + \Gamma_{\mu r}^\mu T^{\mu 0} + \Gamma_{\mu r}^0 T^{\mu r} = 0 \quad \partial_\mu T^{\mu 0} = -\Gamma_{\mu r}^\mu T^{\mu 0} - \Gamma_{\mu r}^0 T^{\mu r}$$

(!) No "strict" energy conservation $\partial_\mu T^{\mu 0} \neq 0$

• Where is that energy gone?!

$\rightarrow$ It is "stored by the space-time"

$\rightarrow$ but... in Einstein eq.s $T_{\mu\nu}^{\text{gravity}}$ is missing! $T_{\mu\nu}^{\text{gravity}}$ is a complicated topic

• Example with gravitational waves (GW)

GW comes from solution of the field equations in vacuum $G_{\mu\nu} = 0$ (linearized)

To look where the energy they transport is gone:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}^{(1)} + h_{\mu\nu}^{(2)} \quad \text{Need to keep at least 2nd order in } h \text{ perturbations} \quad \left(\text{at 1st order } \partial_\mu T^{\mu 0} = 0 \right. \\ \left. \text{no energy transfer to the field} \right)$$

$$G_{\mu\nu}^{(1)}(\eta + h^{(2)}) + G_{\mu\nu}^{(2)}(\eta + h^{(1)}) = 0 \quad G_{\mu\nu}^{(1)}(\eta + h^{(2)}) = \frac{8\pi G}{c^4} t_{\mu\nu} \quad t_{\mu\nu} = -\frac{c^2}{8\pi G} G_{\mu\nu}^{(2)}(\eta + h^{(1)}) \quad \text{not a tensor}$$

$$E = \int_\Sigma t_{00} d^3x \quad \text{total energy on a surface of constant time } \Sigma$$

$$\Delta E = \int_\Sigma t_{0\mu} n^\mu d^3x dt \quad \text{total energy radiated through to infinity}$$

Appendix: explicit computations

- Adiabatic condition (page 3)

$$\nabla_\mu T^{\mu\nu} = 0 \quad T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu + p g^{\mu\nu} \quad \text{for metric compatibility } \nabla_\mu g^{\mu\nu} = 0$$

$$u^\mu u^\nu \nabla_\mu \left(\rho + \frac{p}{c^2}\right) + \left(\rho + \frac{p}{c^2}\right) [u^\nu \nabla_\mu u^\mu + u^\mu \nabla_\mu u^\nu] + g^{\mu\nu} \nabla_\mu p = 0 \quad \xi(t) \quad \nabla_\mu p = 0$$

$$-c^2 u^\mu \nabla_\mu \left(\rho + \frac{p}{c^2}\right) + \left(\rho + \frac{p}{c^2}\right) [-c^2 \nabla_\mu u^\mu + 0] + u_\nu g^{\mu\nu} \nabla_\mu p = 0 \quad \downarrow \times u_\nu$$

$$-u^\mu \nabla_\mu (\rho c^2 + p) - (\rho c^2 + p) \nabla_\mu u^\mu + u^\mu \nabla_\mu p = 0$$

$$u^\mu \delta_\mu (\rho c^2) + (\rho c^2 + p) (\delta_\mu u^\mu + \nabla_\mu u^\mu) = 0 \quad u^\mu = \delta_0^\mu = \delta_0^\mu \quad \nabla_\mu \delta_0^\mu - \nabla_\mu^\mu = \frac{\dot{c}}{c}$$

$$\delta_0 (\rho c^2) + (\rho c^2 + p) (0 + 3 \frac{\dot{c}}{c}) = 0$$

$$\dot{\rho} c^2 + 3 \frac{\dot{c}}{c} (\rho c^2 + p) = 0 \quad \boxed{\dot{\rho} c^2 + 3 H (\rho c^2 + p) = 0}$$

$$\otimes_1 \quad u_\nu \nabla_\mu u^\nu + u^\nu \nabla_\mu u_\nu = 2 u_\nu \nabla_\mu u^\nu = \nabla_\mu (u^\nu u_\nu) = \nabla_\mu (-c^2) = 0$$

In the 1st law of Thermodynamic

$$H = \frac{\dot{c}}{c} \Rightarrow \dot{\rho} c^2 + 3 \frac{\dot{c}}{c} (\rho c^2 + p) = 0 = \underbrace{\dot{\rho} c^3}_{(\cdot \dot{c}^3)} + 3 \dot{c} \rho c^2 + 3 \dot{c} p = \frac{d}{dt} (\rho c^3) + p \frac{d}{dt} c^3 = \frac{dE}{dt} + p dV = 0$$

- Einstein eq.s can be written as:

$$g^{\mu\nu} G_{\mu\nu} = g^{\mu\nu} R_{\mu\nu} - \frac{1}{2} R g^{\mu\nu} g_{\mu\nu} = \kappa g^{\mu\nu} T_{\mu\nu} \quad R - \frac{1}{2} 4R = \kappa T \quad \boxed{R = -\kappa T}$$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = R_{\mu\nu} + \frac{\kappa}{2} T g_{\mu\nu} = T_{\mu\nu} \quad \Rightarrow \quad \boxed{R_{\mu\nu} = \kappa \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right)}$$

- Most general spherical metric in a time slice $t = \text{fixed}$ (rest frame at center of frame)

$$\text{For convenience use: } g_{00} = -e^{2A(r)} \quad g_{rr} = e^{2B(r,t)}$$

$$\boxed{ds^2 = g_{00} c^2 dt^2 + g_{rr} dr^2 + r^2 d\Omega^2 = -e^{2A(r)} c^2 dt^2 + e^{2B(r,t)} dr^2 + r^2 d\Omega^2}$$

o "exp" to have a positive value

o e^2 for convenience (square)

o For from the source $\Rightarrow$ Minkowski metric

$$\lim_{r \rightarrow \infty} g_{00} = -1 \quad \lim_{r \rightarrow \infty} A(r) = 0$$

$$\lim_{r \rightarrow \infty} g_{rr} = 1$$

$$\lim_{r \rightarrow \infty} B(r,t) = 0$$

- Einstein tensor (non zero components) See Schwarzschild solution (Mattei GR lecture notes)
(↳ same isotropy condition of FLRW, $e^{2B(r)} \rightarrow f(r)$)

$$\begin{cases} G_{00} = \frac{1}{r^2} e^{2A} \frac{d}{dr} [r(1-e^{-2B})] \\ G_{rr} = -\frac{1}{r^2} e^{2B} (1-e^{-2B}) + \frac{2}{r} \frac{dA}{dr} \\ G_{\theta\theta} = r^2 e^{-2B} (A'' + A'^2 + \frac{A'}{r} - A'B' - \frac{B'}{r}) \\ G_{\phi\phi} = G_{\theta\theta} \sin^2\theta \end{cases} \quad \frac{d}{dr} = "1"$$

other components = 0

- Constraint on spatial curvature

$$\begin{aligned} G^i_i &= g^{ij} G_{ij} = e^{-2B} G_{rr} + r^{-2} G_{\theta\theta} + r^{-2} \sin^2\theta G_{\phi\phi} \quad i=1,2,3 \\ &= e^{-2B} G_{rr} + 2 r^{-2} G_{\theta\theta} \\ &= -e^{-2B} \left(\frac{1}{r^2} e^{2B} (1-e^{-2B}) + \frac{2}{r} \frac{dA}{dr} \right) + r^{-2} 2 r^2 e^{-2B} \left(A'' + A'^2 + \frac{A'}{r} - A'B' - \frac{B'}{r} \right) \\ &= -\frac{1}{r^2} (1-e^{-2B}) - 2 e^{-2B} \frac{B'}{r} = \boxed{-\frac{1}{r^2} [r(1-e^{-2B})]'} \stackrel{!}{=} 3\Lambda \text{ const.} \end{aligned}$$

G same as in Schwarzschild but not dependency in $g_{00}(t)$

- Christoffel symbols:

$$\begin{aligned} \Gamma_{bc}^a &= \frac{1}{2} g^{ax} (\delta_b g_{xc} + \delta_c g_{bx} - \delta_x g_{bc}) = \frac{1}{2} g^{ax} (\delta_b g_{ac} + \delta_c g_{ba} - \delta_a g_{bc}) \quad i,j=1,2,3 \quad \mu,\nu=0,1,2,3 \\ \Gamma_{ij}^0 &= \frac{1}{2} g^{0x} (\delta_i g_{xj} + \delta_j g_{ix} - \delta_x g_{ij}) = \frac{1}{2} g^{00} (\delta_i g_{0j} + \delta_j g_{i0} - \delta_0 g_{ij}) = +\frac{1}{2} \delta_0 g_{ij} \\ \Gamma_{ij}^0 &= 0 \quad i \neq j \quad \Gamma_{11}^0 = \frac{1}{2} \frac{\delta}{\delta t} \left(\frac{\partial^2}{(1-kr^2)} \right) = \frac{2\partial\dot{a}}{2a(1-kr^2)} \quad \Gamma_{22}^0 = \frac{1}{2} \delta_0 \partial^2 = \frac{\partial\dot{a}}{a} r^2 \quad \Gamma_{33}^0 = \Gamma_{22}^0 \sin^2\theta \\ \Gamma_{11}^1 &= \frac{1}{2} g^{11} (\delta_1 g_{11} + \delta_1 g_{11} - \delta_1 g_{11}) = \frac{1}{2} \frac{(1-kr^2)}{a^2} \frac{\delta}{\delta r} \left[\frac{\partial^2}{(1-kr^2)} \right] = \frac{1}{2} \frac{(1-kr^2)}{a^2} \left[\frac{\partial}{\partial r} \frac{-2kr}{(1-kr^2)^2} \right] = \frac{-kr}{(1-kr^2)} \\ &\dots \end{aligned}$$

- Useful to describe dark matter haloes:

$$\begin{aligned} M_{200} &= 200 \rho_c \frac{4}{3} \pi r_{200}^3 \\ r_{200}^3 &= \frac{3}{800\pi} \frac{M_{200}}{\rho_c} = \frac{3 M_{200}}{800\pi} \frac{1}{\rho_c} = \frac{G}{100 H^2} M_{200} \end{aligned}$$

Cosmological distances

Various definitions depend on how the measure is done:

- To change variable: $z \rightarrow z$ redshift

$$dz \rightarrow dz \quad a = (1+z)^{-1} \quad da = -\frac{dz}{(1+z)^2}$$

$$dt \rightarrow da \quad H \equiv \frac{\dot{a}}{a} \quad \dot{a} = \frac{da}{dt} = -aH(z) \quad dt = \frac{da}{\dot{a}} = \frac{da}{-aH(z)}$$

- Light travel distance: distance covered by a photon in dt along lightcone $c dt$

$$dD_L \equiv c dt = \frac{c da}{a H(z)}$$

$$D_L = \frac{c}{H_0} \int \frac{da}{a^2 E(z)}$$

- Comoving distance: distance between 2 hyper surfaces at fixed t
"grid" comoving with expansion

$$ds^2 = -c^2 dt^2 + a^2 d\Omega^2 = 0$$

$$d\Omega = c \frac{dt}{a} = \frac{c da}{a^2 H(z)} = \frac{dD_C}{a}$$

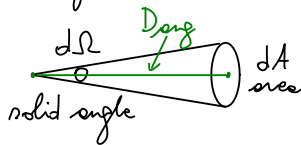
$$D_C = \frac{c}{H_0} \int \frac{da}{a^2 E(z)}$$

analogy to conformal time $d\tau = \frac{dt}{a}$

- Proper distance: "physical" expanding grid

$$D_P(z) = a D_C(z)$$

- Angular diameter distance: distance obtained by measuring angles



$$dA \equiv d\Omega D_{ang}^2$$

$$\Rightarrow D_{ang} = \left(\frac{dA}{d\Omega} \right)^{1/2}$$

solid angle $\rightarrow \frac{d\Omega}{4\pi} = \frac{dA}{4\pi a^2 D_C^2}$ ← area

solid angle of sphere $\rightarrow \frac{d\Omega}{4\pi} = \frac{dA}{4\pi a^2 D_C^2}$ ← area of sphere

$$\Rightarrow \frac{dA}{d\Omega} = a^2 D_C^2$$

$$D_{ang} = a D_C$$

- Luminosity distance: distance obtained by measuring a flux F

$$F \equiv \frac{L}{4\pi D_L^2} \left[\frac{J}{\text{m}^2} \right]$$

redshift, "stretching of λ " : $\left(\frac{\partial_1}{\partial_2} \right)$

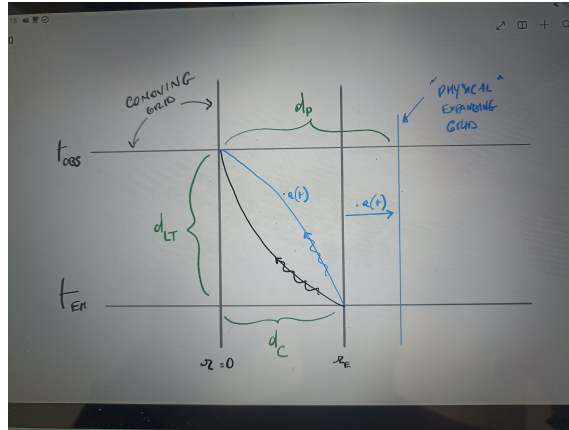
spatial dilution: $\left(\frac{\partial_1}{\partial_2} \right)^2 \Rightarrow \propto \left(\frac{\partial_1}{\partial_2} \right)^4$ on F

delayed arrival time γ_s : $\left(\frac{\partial_1}{\partial_2} \right)$

related to angles

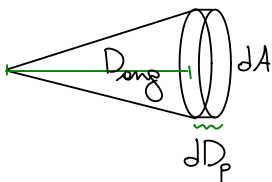
$$D_L = \left(\frac{\partial_1}{\partial_2} \right)^2 D_{ang}(\partial_1, \partial_2)$$

D_{ang} between ∂_1 and ∂_2



Volumes

- Proper volume



$$dV_p = \overset{\text{area}}{dA(z)} dD_p = D_{\text{ang}}^2(z) d\Omega \frac{c dz}{H_0 z E(z)} = D_{\text{ang}}^2(z) \frac{c}{H_0} \frac{dz d\Omega}{(1+z) E(z)}$$

- Comoving volume

$$dV_c(z) = \frac{dV_p(z)}{d^3} = (1+z) dV_p(z) = \frac{c}{H_0} \frac{(1+z)^2 D_A^2(z)}{E(z)} dz d\Omega$$

Why are distances and volumes relevant?

- D_{lum}: observation of standard candles, eg. cepheids, Type 1a Supernovae
- D_{ang}: " " " rulers, eg. CMB, galaxy clustering, Lyman- α forest, lensing, ...
- Number counts: Redshift surveys of galaxies or clusters

Assume a population of objects with comoving density $n_c(z)$

$$\text{Number of objects: } dN = n_c(z) dV_c \Rightarrow \frac{dN}{dz d\Omega} = n_c(z) \frac{dV_c}{dz d\Omega} = n_c(z) \frac{c}{H_0} \frac{(1+z)^2 D_A^2(z)}{E(z)}$$

• structure formation
• geometry

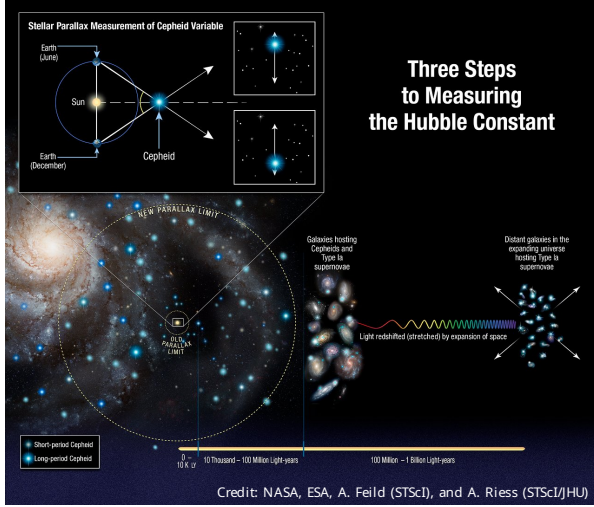
• astrophysics

• selection function (observations)

$\Rightarrow$ Cosmology: knowing a population, you constrain the properties of the universe (geometry)
 eg. non evolving population $n_c(z) = \text{const.} \Rightarrow$ geometry only

Astrophysics: for a fixed cosmology, study evolution of population

Cosmic distance ladder



Luminosity distances

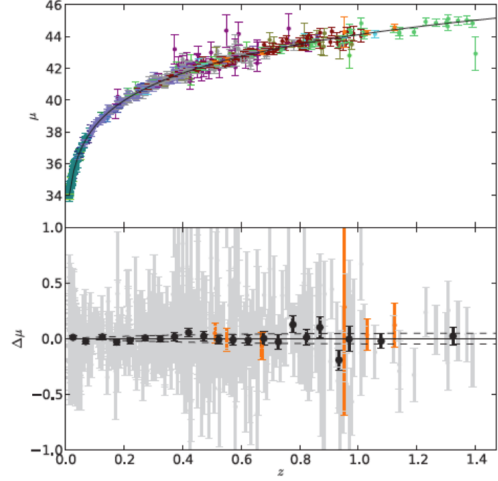


Figure 6.1.2: Union2 combined catalog of 557 SNIa. Here $\mu = 5 \log d_L + \text{const.}$ From Amanullah et al. (2015) (© AAS. Reproduced with permission). Bottom panel: residuals.

Angular diameter distance (gravitational lensing)

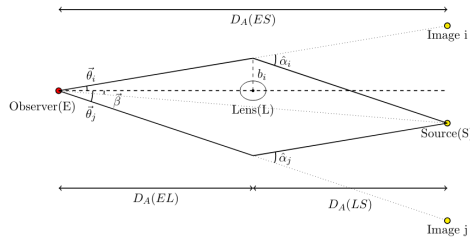


Figure 1: Configuration of a strong lens system, with definition of the variables used throughout this paper. All angles are measured with respect to the center of the lens galaxy; θ is the angular position of the image; β is the angular position of the source in the absence of the lens; α is the scaled deflection angle; $\hat{\alpha}$ is the deflection angle at the lens plane; and b is the physical separation to the closest approach at the lens plane.

Jee, Komatsu, Suyu
(arXiv:1410.7770)

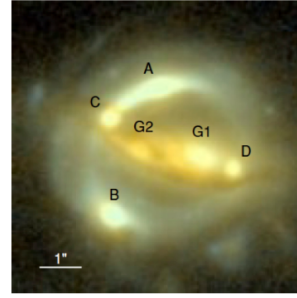


Figure 2: Image of B1608+656, adopted from figure 1 of [7].

Volumes and populations (Galaxy Clusters)

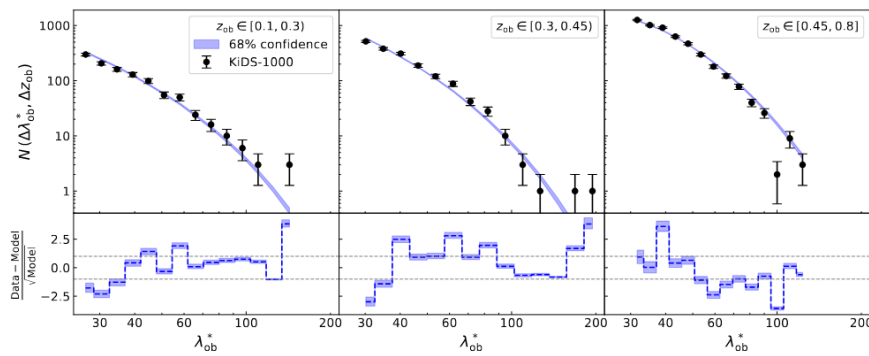


Fig. 10. Top panels: Counts of the AMICO KiDS-1000 galaxy clusters in the redshift bins adopted in the analysis (increasing from left to right). The black dots show the measures, with the error bars corresponding to Poisson uncertainties. The blue bands display the 68% confidence levels of the model, derived from the posterior of all the free parameters considered in the joint analysis of counts and weak lensing. Bottom panels: Pearson residuals. The horizontal dashed grey lines show the interval between -1 and 1.

Maturi et al. (2025)
Lesci et al. (2025)

Cosmological horizons

Particle horizon

$$ds=0 \quad c dt = a d\tau \quad R_c = c \int_0^t \frac{dt'}{a} = \boxed{R_c = \frac{c}{H_0} \int_0^1 \frac{dz'}{a'^2 E(z')}} \quad \text{comoving} \quad \boxed{R = a R_c(z)} \quad \text{proper}$$

- global/integrated property: How far has light travelled since $t=0$
- Depends on the past expansion of the universe
- Causally connected points
- Set furthest object we can see, size of observable universe
- Monotonic growth with time
- $R = 46 \cdot 10^9$ light-years even if $t_H = 13.8 \cdot 10^8$ years (space is expanding)

$$d\tau = dt/a : \text{conformal time}$$

Event horizon

$$R_{e,c} = \frac{c}{H_0} \int_a^\infty \frac{dz'}{a'^2 E(z')} \quad \text{comoving} \quad \boxed{R_e = a R_{e,c}(a)} \quad \text{proper}$$

- maximum distance from which light emitted now can ever reach us in future
- Depends on the future expansion of the universe
- if $\begin{cases} \text{no} & \Rightarrow & \text{not an horizon} & (\text{e.g. decelerated expansion}) \\ \text{convergence} & \Rightarrow & \text{there is an horizon} & (\text{e.g. accelerated expansion}) \end{cases}$ Yes! Λ CDM $R_H \sim 16 \cdot 10^9$ ly
we can wait forever but signal can not reach us

Hubble radius

$$\boxed{R_H \equiv \frac{c}{H(t)}} \quad (\text{Hubble's law } v = a H : a = \frac{v}{H}) \quad R_H = \frac{c}{H_0} = 4,1 \text{ Gpc}$$

- it is not a real horizon
- it ignores expansion history (not an integral function)
- local/instantaneous characteristic scale, where recession speed $= c$, for larger distance $v > c$
- maximum scale over which physical processes can maintain local causal connection at time t
↳ e.g. scale for thermal equilibrium or coherent oscillatory mode
- it can decrease with time (when $\ddot{a} > 0$, accelerated expansion)

Scalar fields in cosmology

Why scalar fields? simple but very flexible, you could also introduce vector or tensorial

Two important scalar fields

- 1) inflaton field (early times) early accelerated expansion
- 2) dark energy (late times) current " "

Simplest model: scalar field $\phi \in \mathbb{R}$

Spin 0 particle like Higgs field

Lagrangian density

$$\mathcal{L} = -\frac{1}{2} c^2 \nabla_\mu \phi \nabla^\mu \phi - U(\phi)$$

energy-momentum

$$T_{\mu\nu} = c^2 \nabla_\mu \phi \nabla_\nu \phi + \mathcal{L} g_{\mu\nu} = c^2 \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} c^2 \nabla_\alpha \phi \nabla^\alpha \phi g_{\mu\nu} - U(\phi) g_{\mu\nu}$$

(-+++) signature

$U(\phi)$: potential energy = amount of energy associated to a certain field value

Self interaction, couplings to other fields (eg. matter), decays/annihilations

$$U(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 - \frac{1}{2} \xi R \phi^2 + \dots \quad m \neq 0 \text{ massive}, \lambda \neq 0 \text{ self-interacting}$$

$\xi \neq 0$ non minimal coupling to R

$g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi$: Kinetic term: it incorporates the dynamics of the scalar field

if absent: E-L eq. $\frac{\delta \mathcal{L}}{\delta \phi} - \nabla_\mu \left(\frac{\delta \mathcal{L}}{\delta \nabla_\mu \phi} \right) = \frac{\delta \mathcal{L}}{\delta \phi} = \frac{\delta V}{\delta \phi} = 0$ not a wave eq.

it can not propagate (no excitation = n particles)

$\Rightarrow$ vacuum (static configuration), ϕ is missing

minimal coupling to gravity; gravity affects ϕ but not vice versa

Note: $\nabla_\alpha \phi = \partial_\alpha \phi$ because $\phi \in \mathbb{R}$

write ∇_α for conceptual clarity (covariance). Recall: $\nabla_\alpha \nabla_\beta \phi \neq \partial_\alpha \partial_\beta \phi$

- Isotropy: only diagonal components of T are $\neq 0$

Homogeneity: $\delta_i \phi = 0$ i.e. $\phi(t)$ only as well as isotropy (vector $\nabla \phi = 0$ = privileged direction)

$$\left\{ \begin{array}{ll} T_{00} = \frac{1}{2} \dot{\phi}^2 + U(\phi) + \frac{1}{2} c^2 (\nabla \phi)^2 \stackrel{!}{=} \rho c^2 & \text{energy density} \\ T_{ii} = \frac{1}{2} \dot{\phi}^2 - U(\phi) - \frac{1}{6} c^2 (\nabla \phi)^2 \stackrel{!}{=} P & \text{pressure} \\ P = w \rho c^2 : w = \frac{P}{\rho c^2} = \frac{\frac{1}{2} \dot{\phi}^2 - U}{\frac{1}{2} \dot{\phi}^2 + U} & \text{effective eq. of state} \end{array} \right. \quad \left. \begin{array}{l} \text{"..."} = \frac{1}{c^2} \frac{d}{dt} \\ \text{cosmological time: } \tau \\ \text{ideal fluid } T_{\mu\nu} = \left(\rho + \frac{P}{c^2} \right) u_\mu u_\nu + P g_{\mu\nu} \end{array} \right\}$$

not constant in general!

- From adiabatic condition

$$\rho = \rho_0 a^{-3(1+w(z))} \quad \text{density evolution might depend on a varying } w(z)$$

Dynamic of the scalar field

- No coupling with matter

- Find evolution of the field, i.e. eq. of motion, as for the other components: Adiabatic evolution

$$\dot{\phi}^2 + 3H(\dot{\phi}^2 + \rho) = 0 \quad \frac{1}{2} \dot{\phi}^2 + \frac{\delta U}{\delta \phi} \dot{\phi} + 3H\left(\frac{1}{2}\dot{\phi}^2 + U(\phi) + \frac{1}{2}\dot{\phi}^2 - U(\phi)\right) = 0 \quad \text{for } U(\phi) = \frac{1}{2}m^2\phi^2 \quad m^2 = m_{\text{physical}}^2$$

$$\Rightarrow \underbrace{\ddot{\phi} + 3H\dot{\phi} + \frac{\delta U}{\delta \phi}}_{\square \phi} = 0 \quad \text{"Klein-Gordon eq." in an expanding universe}$$

damped oscillator, H = "friction" term

mass parameter in Lagrangian $m^2 = \frac{d^2 U}{d\phi^2}$ (bare mass) parameter of the theory

Dynamics depends on U , choose the model of your like

get inspiration from supersymmetry, extra dimensions, ...

In fact KG eq.

$$(\square - m^2)\phi(t, x) = 0 \quad \cdot U = \frac{1}{2}m^2\phi^2 \Rightarrow \frac{\delta U}{\delta \phi} = m^2\phi \quad \leftarrow \text{free = no self interaction}$$

$$\cdot \square \phi = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \phi) = -(3H\dot{\phi} + \ddot{\phi}) + \partial^2 \phi \quad \text{homogeneity } \partial_i \phi = 0$$

$$\mu=0: \partial^3 \partial_\mu (\partial^3 g^{\mu\nu} \partial_\nu \phi) = -\partial^3 (3\partial^2 \dot{\phi} + \partial^3 \ddot{\phi}) = -(3H\dot{\phi} + \ddot{\phi})$$

$$\mu=i: \partial^3 \partial_i (\partial^j g^{ij} \partial_j \phi) = \partial_i \partial^2 \dot{\phi} = \partial^2 \nabla^2 \phi$$

FLRW: $g^{\mu\nu} = -1, \sqrt{-g} = a^3$ bare trace not $g^{\mu\nu}$

only t dependency

Total eq. of motion

$$\begin{cases} \ddot{\phi} + 3H\dot{\phi} + \frac{\delta U}{\delta \phi} = 0 \\ H^2 = \frac{8\pi G}{3} \rho - \frac{\Lambda}{3} = \frac{8\pi G}{3c^2} \left(\rho_{\text{rad}} c^2 + \frac{1}{2}\dot{\phi}^2 + U(\phi) \right) - \frac{\Lambda c^2}{3} \end{cases}$$

$$\rho_{\text{rad}} = \text{radiation}, m$$

ϕ affects expansion,
expansion affects ϕ

For our 2 cases: $\begin{cases} \text{DE substitutes } \Lambda & \Rightarrow \Lambda \neq 0 \\ \text{At early times } \Lambda \text{ is negligible} & \Rightarrow \Lambda \approx 0 \end{cases}$

What about $U(\phi)$?

- Pick up your poison many models are out there, eg. $U(\phi) = \frac{1}{2}m\phi^2$

- For examples, see Appendix

- To investigate qualitative behaviour of solutions, use phase-space analysis (see advanced cosmology notes)

One model, different physical scenarios

we $\rho_\phi^2 = \frac{1}{2} \dot{\phi}^2 + U(\phi)$ $p_\phi = \frac{1}{2} \dot{\phi}^2 - U(\phi)$ $w_\phi = \frac{p_\phi}{\rho_\phi}$ $\rho = \rho_0 a^{-3(1+w)}$

A) Steep potential: $\frac{1}{2} \dot{\phi}^2 \gg U(\phi)$

$w_\phi \simeq 1$, $\rho_\phi \propto a^{-6}$ it dies faster than the rest

$U=0$: massless scalar field $m^2 = U' = 0$ (stiff matter)



B) Kinetic energy = potential energy: $\frac{1}{2} \dot{\phi}^2 = U$

ϕ oscillates harmonically around a zero-energy potential

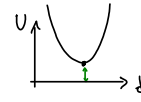
$w_\phi = 0$, $\rho_\phi \propto a^{-3}$ $\Leftarrow$ averaging over several cycles $\langle p \rangle = 0$ Dust!



C) Static field: $\dot{\phi} = 0$ at non-zero energy minimum of U

$w_\phi = -1$, $\rho_\phi = \frac{U}{c^2} > 0$ field represents the cosmological const. Λ !

$\neq 0$ or $\rho_\phi^2 = U = 0$ = "nothing" state in which the field has non zero vacuum energy



D) Slow roll limit $\dot{\phi}^2 \ll U(\phi)$

• Dynamical field ($\dot{\phi} \neq 0$), slow evolution, no oscillations

• $w_\phi = \frac{p}{\rho} = \frac{\frac{1}{2} \dot{\phi}^2 - U}{\frac{1}{2} \dot{\phi}^2 + U} = \frac{-U(1 - \frac{1}{2} \frac{\dot{\phi}^2}{U})}{U(1 + \frac{1}{2} \frac{\dot{\phi}^2}{U})} \simeq -\left(1 - \frac{1}{2} \frac{\dot{\phi}^2}{U}\right) \simeq -1 + \frac{\dot{\phi}^2}{U} \Rightarrow w_{\text{eff}} \text{ slightly larger than } -1!$



necessary dynamic! as field evolves $\dot{\phi} \uparrow \Rightarrow w \uparrow \Rightarrow$ key difference from Λ

Constraint on w_ϕ : $w_\phi + 1 = \frac{p + \rho c^2}{\rho c^2} = \frac{\dot{\phi}^2}{\rho c^2} > 0$ for physically reasonable systems $\Rightarrow w > -1$ (!) $\neq -1$

• for $w_\phi < -1$ most theories are unstable: ghosts (No lower energy bound: system can produce ∞ energy)

• slow roll limit ensure a long phase of accelerated expansion
"long" relative to the other processes taking place at a certain cosmic epoch

Condition on w for accelerated expansion

$\ddot{a} = -\frac{4\pi G}{3}(\rho + \frac{3p}{c^2}) + \frac{\Lambda}{3} > 0 \Rightarrow (\rho + \frac{3p}{c^2}) < 0 \Rightarrow p < -\frac{1}{3} \rho c^2 \Rightarrow w < -\frac{1}{3} \Rightarrow -1 \leq w \leq -\frac{1}{3}$

$\Rightarrow$ Slow roll condition fulfills this criteria as $w_\phi = -1 + \epsilon$ $0 < \epsilon \ll 1$

$\Rightarrow$ case (D) can be used for DE and inflation

• As $\dot{\phi}$ evolves: slow roll condition is broken $\rightarrow$ start oscillations $\rightarrow$ Hubble friction, $3H$, becomes relevant
 $\rightarrow$ accelerated expansion ends: relevant for inflation

Appendix

Potentials

• Free field (trivial vacuum)

$$U(\phi) = \frac{1}{2} m^2 \phi^2$$

minimum is at $\phi_{vac} = 0 \Rightarrow m_{phys}^2 = U''(\phi)|_{\phi_{vac}} = m^2$ physical mass = bare mass
 eq. of motion: $(\square + m^2)\phi = 0$
 spin-0 particle, simple "toy model"
 $m=0$ massless (only $\partial_\mu \phi \partial^\mu \phi$ term)

• Self-interacting

$$U(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4$$

$\phi_{vac} = 0$, $m_{phys} = m$
 λ = coupling const. how strongly the field interact with itself
 it produces scattering processes such as $\phi + \phi \rightarrow \phi + \phi$
 eq. of motion: $\square \phi + m^2 \phi + \frac{\lambda}{3!} \phi^3 = 0$
 eg. dilaton field, pion in chiral limit (quark masses $\rightarrow 0$)
 Goldstone bosons ($m=0$) in spontaneous symmetry breaking, some inflaton

• Symmetry breaking (Mexican hat)

$$U(\phi) = -\frac{1}{2} \mu^2 \phi^2 + \frac{\lambda}{4!} \phi^4$$

$U'(\phi) = -\mu^2 \phi + \lambda \phi^3 \stackrel{!}{=} 0 \Rightarrow \phi_{min} = \pm \mu/\sqrt{\lambda} = \phi_{vac} \neq 0$ Shifted vacuum
 $U''(\phi) = -\mu^2 + 3\lambda \phi^2 \Rightarrow m_{phys}^2 = U''|_{\phi_{vac}} = -\mu^2 + 3\lambda \mu^2/\lambda = 2\mu^2$ mass depends on background value
 Higgs boson after electroweak symmetry breaking

• General potential

$$U(\phi) = C_0 + C_1 \phi + \frac{1}{2} C_2 \phi^2 + \frac{1}{3!} C_3 \phi^3 + \frac{1}{4!} C_4 \phi^4 + \dots$$

ϕ around minima

the physically allowed terms depend on the symmetry of the system

symmetries: eg. $\phi \rightarrow -\phi$, $V(\phi) = V(-\phi)$ $\mathbb{Z}_2$ (discrete) symmetry
 even powers ($\phi^2, \phi^4, \dots$) remain the same $\Rightarrow$ respect $\mathbb{Z}_2$ sym.
 odd " change sign $\Rightarrow$ do not respect $\mathbb{Z}_2$ sym.

with even power: symmetric potential, it does not matter on which side you roll

$\Rightarrow$ symmetric self interaction, interaction vertices:

$\phi^4 \rightarrow$ 4-point vertex, elastic 2-body scattering 
 $\phi^6 \rightarrow$ 6-point vertex, ...

with odd powers: non symmetric ("tilted"), one side might be deeper than another

$\Rightarrow$ different behaviour on the two sides of the potential, no above symmetry

$\Rightarrow$ asymmetric self interaction, interaction vertices:

$\phi^3 \rightarrow$ 3 vertices, a particle can split in 2 or merge from 2

$\phi^5 \rightarrow$ 5 "

allows creation, annihilation

ϕ_{vac} might not be zero

In 4-D, allowed terms go up to Φ^4 Dimensional analysis

$\hbar = c = k_B = 1$ energy, mass, momentum, length⁻¹, time⁻¹ have the same units

$[S] = 0$ Action is dimensionless $[S] = \frac{J}{c}$

$[dx^4] = \text{length}^4$ $[\mathcal{L}] = \text{length}^{-4} \cdot \text{mass}^4 \rightarrow [\mathcal{L}] = 4$ recall: $S = \int \mathcal{L} dx^4$ (4-D theory)

$\Rightarrow$ terms in Lagrangian must be mass⁴

$$\mathcal{L} = -\frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - U(\Phi) \quad U(\Phi) = \sum_{n=0}^N \frac{c_n}{n!} \Phi^n$$

$$[\partial_\mu] = 0 \quad [\partial_\mu] = 1 \quad [\partial_\mu \partial^\mu \Phi] = 4 \quad [\Phi] = 1 \quad [c_n] = 0 \text{ dimensional}$$

$$[c_n \Phi^n] = [c_n] + n[\Phi] \stackrel{!}{=} 4 \Rightarrow [c_n] = 4 - n \Rightarrow n \leq 4!$$

you could have $[c_n] > 0$ but this would break scale invariance

e.g. non minimal coupling to curvature: $[R] = 2 \rightarrow \Phi^2: \xi R \Phi^2$

Renormalizability

A theory is renormalizable if $[c_n] \geq 0 \quad \forall n$
 $\Rightarrow 4 - n \geq 0 \Rightarrow n \leq 4 \Rightarrow \Phi^0, \Phi^1, \Phi^2, \Phi^3, \Phi^4$ are allowed

Strong argument in setting up models

e.g. Scalar field with curvature coupling $\mathcal{L} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} m^2 \Phi^2 - \frac{1}{2} \xi R \Phi^2$
 $R = \text{Ricci scalar}$ (non minimal coupling)
 $\xi = \text{dimensionless coupling constant } \Phi \leftrightarrow R$

Natural choice from dimensional arguments: $[S] = 0, [R] = 2 \Rightarrow [\Phi^2] = 2$ only choice

Causality of $\Phi(\vec{x}, t) = \Phi(t)$

- The same value Φ will change simultaneously across the entire Universe, even ignoring possible horizons
- Does this break causality? No:

a) the system just obeys its own symmetries, here: spatial homogeneity and isotropy

Translations: $\Phi(\vec{x} + \vec{a}, t) = \Phi(\vec{x}, t)$

Rotations: $\Phi(R\vec{x}, t) = \Phi(\vec{x}, t)$ 3D Galilean group $E(3)$

$\hookrightarrow$ It can be an imposed symmetry: assumption

$\hookrightarrow$ " " " a physical " : resulting from a spontaneous symmetry where the system naturally evolves and reach that configuration

e.g. inflation bring to homogeneity (thermal equilibrium region is expanded to cover all scales $< R_H$)

b) $\Phi(t)$ is just a global degree of freedom, same as for the expansion factor, $a(t)$

c) there is no information transfer, not a local "signal"

If $\delta\Phi \neq 0 \Rightarrow$ perturbations travel with $c_s \leq c$ (they are a "signal")

Ghosts: example

$$\mathcal{L} = +\frac{1}{2}c^2\delta_\mu\delta^\mu\dot{\phi}^2 - U(\phi) \quad T_{\mu\nu} = c^2\delta_\mu\delta_\nu\dot{\phi} + \mathcal{L}\delta_{\mu\nu} = c^2\delta_\mu\delta_\nu\dot{\phi} + \frac{1}{2}c^2\delta_\mu\delta^\mu\dot{\phi}^2\delta_{\mu\nu} - U(\phi)\delta_{\mu\nu}$$

$$T_{00} = -\frac{1}{2}\dot{\phi}^2 + U(\phi) \stackrel{!}{=} \rho c^2 \quad *$$

$$T_{ii} = -\frac{1}{2}\dot{\phi}^2 - U(\phi) \stackrel{!}{=} p_\phi$$

$$p = w\rho c^2 : w = \frac{p}{\rho c^2} = \frac{-\frac{1}{2}\dot{\phi}^2 - U}{-\frac{1}{2}\dot{\phi}^2 + U} \quad w+1 = \frac{p+\rho c^2}{\rho c^2} = \frac{-\dot{\phi}^2}{\rho c^2} < 0 \quad \text{if } \dot{\phi}^2 > 0 \Rightarrow w < -1 \quad \checkmark$$

* Hamiltonian is unbound below $\leftrightarrow$ no ground state

catastrophically unstable

runaway production of particles (= energy), quantum: they violate unitarity

cosmologically, $w < -1 \rightarrow$ big rip simple scalar field

• Counter example: $\mathcal{L} = -\frac{1}{2}c^2\delta_\mu\delta^\mu\dot{\phi}^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{3!}\xi\phi^3$

does not produce ghosts, this is just an unstable theory: ϕ rolls to arbitrary large values

Energy-momentum tensor of a scalar field

signature $(-1, 1, 1, 1)$

$$\mathcal{L} = -\frac{1}{2}c^2\delta_\mu\delta^\mu\dot{\phi}^2 - U(\phi) \quad T_{\mu\nu} = c^2\delta_\mu\delta_\nu\dot{\phi} + \mathcal{L}\delta_{\mu\nu} = c^2\delta_\mu\delta_\nu\dot{\phi} - \frac{1}{2}c^2\delta_\mu\delta^\mu\dot{\phi}^2\delta_{\mu\nu} - U(\phi)\delta_{\mu\nu}$$

$$T_{00} = c^2(\delta_0\dot{\phi})^2 + \frac{1}{2}c^2[-(\delta_0\dot{\phi})^2 + (\nabla\dot{\phi})^2] - U(\phi)(-1) = \dot{\phi}^2 - \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}c^2(\nabla\dot{\phi})^2 + U(\phi) = \frac{1}{2}\dot{\phi}^2 + U(\phi) + \frac{1}{2}c^2(\nabla\dot{\phi})^2$$

$$T_{ii} = c^2(\delta_i\dot{\phi})^2 - \frac{1}{2}c^2[-(\delta_0\dot{\phi})^2 + (\nabla\dot{\phi})^2] \cdot 1 - U(\phi) \cdot 1 = c^2(\delta_i\dot{\phi})^2 + \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}c^2(\nabla\dot{\phi})^2 - U(\phi) = \frac{1}{2}\dot{\phi}^2 - U(\phi) - c^2\frac{1}{6}(\nabla\dot{\phi})^2$$

$$* (\nabla\dot{\phi})^2 = \nabla\dot{\phi} \cdot \nabla\dot{\phi} = (\delta_1\dot{\phi})^2 + (\delta_2\dot{\phi})^2 + (\delta_3\dot{\phi})^2 = 3(\delta_i\dot{\phi})^2$$

here we defined " $\dot{\cdot}$ " $\equiv \frac{d}{cd\tau}$ dot " $\cdot$ " derivative with respect to conformal time $\frac{d}{dx^0} = \frac{d}{cdt} \equiv \frac{d}{cd\tau}$

Recall: $ds^2 = \dot{\alpha}^2(t)[-c^2d\tau^2 + d\chi^2 + f_\chi^2(\chi)(d\vartheta^2 + \sin\vartheta d\phi^2)]$ $d\tau \equiv \frac{dt}{a(t)}$ $\tau = \text{conformal time}$

Appendix

Vacuum

Determines the background and physical properties (mass, symmetries, ...) of associated particle

Classical: $\phi_{vac} = \phi_0$ vacuum expectation value: field value at minimum of $U(\phi)$ $\left. \frac{dU}{d\phi} \right|_{\phi_0} = 0$

"rest position" of the field, with no disturbances, it remains there
classically there are no particles

Mass of a scalar field

$$(x^\mu) = (t, \vec{x}) \quad (k^\mu) = (\omega, \vec{k})$$

1) Free-field limit

$$(\Box + m^2)\phi = 0$$

$$\Box = \partial_\mu \partial^\mu = -\eta^{\mu\nu} \partial_\mu \partial_\nu = -\partial_0^2 + \vec{\nabla}^2 \quad \text{free field limit}$$

$$\phi = \hat{\phi} e^{ik_\mu x^\mu} = \hat{\phi} e^{i(\vec{k}\vec{x} - \omega t)}$$

plane wave solution (Fourier space)

$$p_\mu p^\mu = -m^2 \Rightarrow$$

$$\text{dispersion relation} \quad E^2 = p^2 + m^2$$

m = rest mass of a single quantum excitation of the field (particle interpretation)
physical mass, what you can observe in an experiment

2) If Field sitting on background

$\phi(x) = \phi_{vac} + \eta$ expand around the equilibrium point, η small perturbation

$$U(\phi) = U(\phi_{vac}) + \frac{1}{2} U''(\phi_{vac}) \eta^2 + \frac{1}{3!} U'''(\phi_{vac}) \eta^3 + \dots \quad \leftarrow U(\phi_{vac}) = 0 \quad '1' = \partial/\partial\phi$$

$$\mathcal{L} = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - U(\phi) = -\frac{1}{2} \partial_\mu \eta \partial^\mu \eta - \frac{1}{2} U''(\phi_{vac}) \eta^2 - \frac{1}{3!} U'''(\phi_{vac}) \eta^3 - \dots \quad -U(\phi_{vac}) \text{ irrelevant for } \delta\mathcal{L} = 0$$

$$\delta\mathcal{L} = 0 \Rightarrow [\Box + U''(\phi)]\phi = 0 \Leftrightarrow (\Box + m^2)\phi = 0 \quad \text{Klein-Gordon eq, free scalar field "mass-spring system"}$$

$$m_{phys}^2 = U''(\phi_{vac}) \quad \text{physical (renormalized) mass, same as at (1)}$$

for $U''(\phi_0) < 0$ exponential growth (instability)

$U \rightarrow U_{eff}$ = effective potential (classical + quantum effects)

There are classical masses only: need to use U_{eff} to account for quantum effects

Dynamical Dark Energy

- Λ is allowed by GR
explains very well observations (accelerated expansion, geometry)
associated to vacuum energy ($T^{\mu\nu}$ has no dependency on u^ν)
- But... if this is so, we get problems..

Planck mass $m_p = \sqrt{\frac{\hbar c}{G}} = 10^{19} \text{ GeV}$

Problem 1a: fine tuning problem

from cosmological observations: $\Omega_\Lambda \approx 0,7$; $H_0 \approx 70 \text{ km s}^{-1} \text{ Mpc}^{-1}$

$$-\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c} = \frac{\rho_\Lambda}{\rho_c} \frac{8\pi G}{3H_0^2} \Rightarrow \rho_\Lambda = \Omega_\Lambda \frac{8\pi G}{3H_0^2} \approx 10^{-47} \text{ GeV}^4 \approx 10^{-123} m_p^4 \quad m_p = 1,22 \cdot 10^{19} \text{ GeV}$$

$\hookrightarrow$ fine tuning problem, why so "unnaturally small" but not zero?!

Problem 1b: QFT point of view, an illustration *free field, Minkowski!*

From an elementary particle perspective: "vacuum energy is $\sim 10^{121}$ times larger!" Not really...

- vacuum energy $\langle \rho \rangle$ of empty space ($t=1=c$)
- zero energy of a field of mass m with momentum k and frequency ω $E = \frac{\omega}{2} = \sqrt{k^2 + m^2}/2$
- sum zero-point energies of the field up to a cut-off $k_{\max} (\gg m)$

$$\rho_{\text{vac}} = \int_0^{k_{\max}} \frac{d^3k}{(2\pi)^3} \frac{1}{2} \sqrt{k^2 + m^2} = \int_0^{k_{\max}} \frac{2\pi k^2 dk}{(2\pi)^3} \frac{1}{2} \sqrt{k^2 + m^2} \approx \int_0^{k_{\max}} \frac{2\pi k^3 dk}{(2\pi)^3 2} = \frac{k_{\max}^4}{16\pi^2} \approx 10^{24} \text{ GeV}^4$$

$\uparrow$ on k shells (isotropy) $\uparrow$ $k \gg m$ dominates $\uparrow$ GR valid up to Planck scale $\Rightarrow$ take $k_{\max}$ to l_p, m_p

Even using other energy scales the problem remains: QCD scale $k_{\max} \approx 0,1 \text{ GeV}$ v.s. $\rho_{\text{vac}} \approx 10^{-3} \text{ GeV}^4$

to be renormalized: cosmology tells you the value of ρ_{vac} (GR sets an absolute scale)

Smallness QCD vacuum $\sim (0,3 \text{ GeV})^4$ Electroweak scale $\sim (100 \text{ GeV})^4$ Planck scale $\sim (10^{18} \text{ GeV})^4$
Observed $\sim (10^{-3} \text{ eV})^4 \Rightarrow$ Fine tuning problem

Problem 2: Coincidence problem

Λ and matter are not linked but $\Omega_{\Lambda,0} \sim \Omega_{m,0}$, why?!

Coincidence time z_c : $\Omega_\Lambda \stackrel{!}{=} \Omega_{m,0} (1+z_c)^3$ $z_c = \left(\frac{\Omega_\Lambda}{\Omega_{m,0}} \right)^{1/3} - 1 = \left(\frac{\Omega_\Lambda}{1-\Omega_\Lambda} \right)^{1/3} - 1 \approx 0,3$ (*flat geometry*)

Side note: $m = \sqrt{U''} \ll H_0 \sim 10^{-33} \text{ eV}$ ultra light particle

Possible solution/alternative models

Anthropic principle

- life requires certain energy scales and sufficient time to form
- large ρ_{DE} are statistically more likely but an excessive expansion would excessively dump structure formation (to little time for us)
- $\rho_{DE} \sim \rho_m$ is the largest ρ_{DE} value allowing a sufficient formation of structure needed to allow life
- We see the universe with these ρ_i values just because this is the configuration allowing life

Back reaction

perturbations exist $\Rightarrow$ they might have an effect on the background expansion

Average over a volume domain: *what we did* $\rightarrow G_{\mu\nu}(\langle g_{\mu\nu} \rangle) \neq \langle G_{\mu\nu} \rangle$ *what we should have done*

$$\frac{\ddot{a}_0}{a_0} = -\frac{4\pi G}{3} \langle \rho \rangle_0 + Q_0 \quad \text{kinematical backreaction}$$

Inhomogeneous models

eg. LTB/ Bondi models, still require strong fine-tuning

Dark-Energy

- 1) $\Lambda \neq 0 \Rightarrow \exists$ of some symmetry (supersymmetric theories)
even if symmetry is broken, still possible to have $\Lambda = 0$

- 2) introduce a new dynamical field: Dark-Energy (DE)

not a constant, it evolves with time: $\rho(z), w(z)$

its behaviour must be very similar to Λ : so far observations are compatible with Λ !

simplest model: scalar field + slow roll condition (case D)

$\Rightarrow$ solves conflict with vacuum energy, but not "smallness" and coincidence... but $\exists$ models with:

Tracker solutions: $\rho_{DE} \rightarrow \rho_m$ regardless the initial conditions

Scaling attractors: have $w_{DE}(t)$ such that $\Omega_m \sim \Omega_{DE}$ at all times and $w_{DE}(t) \rightarrow -1$ at late time
see advanced cosmology....

Here, I will present the simplest case: Quintessence

Simplist model: Quintessence

Self interactive scalar field $\mathcal{L} = -\frac{1}{2}c^2\delta_\mu\phi\delta^\mu\phi - U(\phi)$ $T_{\mu\nu} = c^2\delta_\mu\phi\delta_\nu\phi + \mathcal{L}g_{\mu\nu}$ $\nabla_\mu T^{\mu\nu} = 0$

Homogeneity and isotropy: $\delta_i\phi = 0$ * i.e. no gradients, same value everywhere $\Rightarrow \phi(t)$ only

Must give expansion similar to Λ CDM $\Rightarrow$ Slow roll condition $\dot{\phi}^2 \ll U(\phi)$, case (D)

$$\left\{ \begin{array}{l} T_{00} = \frac{1}{2}\dot{\phi}^2 + U(\phi) = \rho \\ T_{ii} = \frac{1}{2}\dot{\phi}^2 - U(\phi) = p \\ w = \frac{p}{\rho} = \frac{\frac{1}{2}\dot{\phi}^2 - U}{\frac{1}{2}\dot{\phi}^2 + U} \approx -1 + \frac{\dot{\phi}^2}{U} \end{array} \right. \quad \begin{array}{l} \text{energy density} \\ \text{pressure} \end{array} \quad \left. \begin{array}{l} \text{ideal fluid} \\ \text{eq. of state parameter, } \dot{\phi}^2 \ll U \end{array} \right\} \quad \begin{array}{l} \text{conformal time } \rightarrow \\ T_{\mu\nu} = (\rho + \frac{p}{c^2})u_\mu u_\nu + pg_{\mu\nu} \quad " " = \frac{1}{c^2} \frac{d}{dt} \\ w_{de} = -1 + \epsilon \quad 0 < \epsilon \ll 1 \\ -1 < w_{de} \leq -\frac{1}{3} \quad \text{for acceleration} \end{array}$$

$$\left\{ \begin{array}{l} \ddot{\phi} + 3H\dot{\phi} + \frac{\delta U}{\delta\phi} = 0 \\ H^2 = \frac{8\pi G}{3c^2} \left(\rho_\mu c^2 + \frac{1}{2}\dot{\phi}^2 + U(\phi) \right) - \frac{\kappa c^2}{2} \end{array} \right. \quad \left. \begin{array}{l} \text{Klein Gordon eq.} \\ \text{Friedmann eq.} \end{array} \right\} \quad \text{dynamic, eq. of motion: } \rho, p, w, H \text{ evolve}$$

Note: key differences from Λ : $w_{de}(t)$ evolves as a function of time! $w_{de} \neq -1$!!

Note: this simple model does not solve all problems, e.g. coincidence

see advanced cosmology notes for more advanced models with tracking solutions/scaling attractors

Condition on w for accelerated expansion

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left(\rho + \frac{3p}{c^2} \right) + \frac{1}{3} \frac{\dot{\rho}}{\rho} > 0 \Rightarrow \left(\rho + \frac{3p}{c^2} \right) < 0 \Rightarrow p < -\frac{1}{3}\rho c^2 \Rightarrow w < -\frac{1}{3} \Rightarrow -1 \leq w \leq -\frac{1}{3}$$

$\Rightarrow$ Slow roll condition fulfills this criteria as $w_\phi = -1 + \epsilon \quad 0 < \epsilon \ll 1$

Constraint on the scalar field mass

slow roll = ϕ changes significantly only over times $\sim H^{-1}$ *

compare time scales $\phi(t) = \phi_0 + \delta\phi(t) \quad U(\phi) \approx U'(\phi_0) + U''(\phi_0)(\phi - \phi_0) \quad \ddot{\phi} + \dot{\phi} + 3H(\dot{\phi}_0 + \delta\dot{\phi}) + U''\delta\phi = 0$

$\delta\ddot{\phi} + 3H\delta\dot{\phi} + U''\delta\phi = 0$ natural frequency $\omega^2 = m_\phi^2 = U''(\phi_0)$ effective mass squared

slow roll = no oscillations: $U''(\phi) \ll H^2 \Rightarrow U''(\phi) = m_\phi^2 \ll H^2$ * $H_0 \sim 10^{-33} \text{ eV} \Rightarrow m_\phi \lesssim 10^{-33} \text{ eV}$

DE in the Friedmann equations

Condition on w to have acceleration

$$\ddot{a} = -\frac{4\pi G}{3}(\rho + \frac{3p}{c^2}) + \frac{\dot{a}}{a} > 0 \Rightarrow (\rho + \frac{3p}{c^2}) < 0 \Rightarrow \rho > -\frac{1}{3}\rho c^2 \Rightarrow w < -\frac{1}{3} \quad \text{and} \quad \dot{\phi}^2 < V(\phi)$$

DE $\boxed{-1 \leq w \leq -\frac{1}{3}}$ for $w < -1$ most fields are unstable: ghosts

Density ρ_ϕ evolution

Adiabatic condition $\dot{\rho}c^2 + 3H(\rho c^2 + p) = \dot{\rho}c^2 + 3H\rho c^2(1+w(a)) = 0$

$$\Rightarrow \frac{d\rho}{\rho} = -3(1+w(a)) \frac{da}{a} \quad \ln \frac{\rho}{\rho_0} = -3 \int_1^a (1+w(a)) \frac{da}{a}$$

$$\rho = \rho_0 \bar{a}^{-3 \int_1^a (1+w(a)) \frac{da}{a}}$$

Friedmann eq.

$$H^2(a) = H_0^2 \left[\Omega_m \bar{a}^{-3} + \Omega_k \bar{a}^{-2} + \Omega_{de} \bar{a}^{-3 \int_1^a (1+w(a)) \frac{da}{a}} \right]$$

Chavira-Polarski-Linder parametrization

There are many DE models and $w(a)$ evolves slowly over a large range of cosmic time

$\Rightarrow$ Assume linear model for w_{de} : $\boxed{w_{de}(a) = w_0 + w_s(1-a)} = w_0 + w_s \frac{a}{1+a}$

$w_0 = w_{de}(1)$ today

i.e. as Taylor expansion of $w(a)$ at 1st order

$$\Rightarrow d\rho = -3(1+w) \frac{d\bar{a}}{\bar{a}} = -3(1+w_0 + w_s(1-a)) \frac{d\bar{a}}{\bar{a}} \quad \text{integrate}$$

$$\int \frac{d\rho}{\rho} = -3(1+w_0+w_s) \int \frac{d\bar{a}}{\bar{a}} + 3 \int w_s \bar{a} \quad \ln \rho \Big|_1^a = -3(1+w_0+w_s) \ln \bar{a} \Big|_1^a + 3w_s \bar{a} \Big|_1^a$$

$$\ln \rho - \ln \rho_0 = +3(1+w_0+w_s) \ln \bar{a} + 3w_s \bar{a} \quad \Rightarrow \quad \boxed{\rho_{de} = \rho_0 \bar{a}^{-3(1+w_0+w_s)} e^{3w_s \bar{a} - 3w_s}}$$

Friedmann eq.

$$H^2(a) = H_0^2 \left[\Omega_m \bar{a}^{-3} + \Omega_k \bar{a}^{-2} + \Omega_{de} \bar{a}^{-3(1+w_0+w_s)} e^{3w_s \bar{a} - 3w_s} \right]$$

Λ dominated universe: $a \propto \exp(\sqrt{\Omega_\Lambda} H_0 t)$ (de-Sitter model)

Other Dark Energy models

Clustering of DE

In general, a scalar field can be inhomogeneous

- Drop homogeneity assumption $\Rightarrow \nabla_i \phi \neq 0$

($C=1$) K-G eq: $\Box \phi - U'(\phi) = 0$ FLRW $\ddot{\phi} + 3H\dot{\phi} - \frac{1}{a^2} \nabla^2 \phi + U'(\phi) = 0$

"1" $= \frac{d}{d\phi}$

$$\Box \phi = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \phi) = -(3H\dot{\phi} + \ddot{\phi}) + \frac{1}{a^2} \nabla^2 \phi$$

$g = -1 \dot{a}^2 \dot{a}^2 \dot{a}^2 = -a^6$

$\mu=0: \frac{1}{a^3} \partial_0 (a^3 g^{00} \partial_0 \phi) = -\frac{1}{a^3} (3\dot{a}^2 \dot{\phi} + \dot{a}^3 \ddot{\phi}) = -(3H\dot{\phi} + \ddot{\phi})$

$\mu=i: \frac{1}{a^3} \partial_i (a^3 g^{ii} \partial_i \phi) = \partial_i \frac{1}{a^2} \partial_i \phi = \frac{1}{a^2} \nabla^2 \phi$ inhomogeneity $\partial_i \phi \neq 0$
only t dependency

- Assume small perturbations $\delta\phi$, linearize

perturbative approach: $\phi(\vec{x}, t) = \bar{\phi}(t) + \delta\phi(\vec{x}, t)$ $|\delta\phi| \ll \bar{\phi}$, $\delta\phi$ = perturbation

at 1st order: $\ddot{\delta\phi} + 3H\dot{\delta\phi} - \frac{1}{a^2} \nabla^2 \delta\phi + U'(\phi) \delta\phi = 0$

for simplicity Minkowski limit: $\delta\ddot{\phi} - \nabla^2 \delta\phi + U'(\bar{\phi}) \delta\phi = 0$ relativistic wave eq. eg. $U = \frac{1}{2} m^2 \phi^2$, $U' = m^2$ effective mass
 $H \rightarrow 0, a=1$

plug ansatz: $\delta\phi = e^{i(\vec{k}\vec{x} - \omega t)} \Rightarrow -\omega^2 + k^2 + m^2 = 0$ $\omega^2 = k^2 + m^2$ dispersion relation

group velocity: $v_g = \frac{d\omega}{dk} = \frac{k}{\sqrt{k^2 + m^2}} \rightarrow 1 \text{ if } k \gg m$ small scales or massless (flat potential)
 $\rightarrow 0 \text{ if } k \ll m$ large scales or massive field

- Quintessence can not cluster

(1): $C_s < 1 \Rightarrow$ Perturbations can arise $\rightarrow$ clustering, but for $k \ll m$ rapid oscillations = Dust $w=0$ (not DE)

(2): $C_s = 1 \Rightarrow$ No perturbations $\Rightarrow$ No clustering! $\Rightarrow$ quintessence remains smooth

- Constraints on m^2

- No evidence of DE clustering $\Rightarrow$ suggesting small mass, m

- Slow roll condition: $U'' \ll H^2$

$\hookrightarrow m^2 = U''$ should be low to satisfy the slow roll condition $\} \Rightarrow$ suggesting small mass
 $\hookrightarrow m^2 \ll H^2 \Rightarrow$ friction suppresses clustering

Are there clustering DE models?

A) Non-canonical field (k-essence)

$\mathcal{L} = K(\phi, X)$ with K an arbitrary function of ϕ and $X = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi$ (kinetic term) canonical eg. $K(\phi, X) = X - U(\phi)$

$\hookrightarrow T^{\mu\nu}: \quad \rho = 2XK_{,X} - K, \quad p = K(\phi, X)$

$\hookrightarrow C_s^2 = \frac{K_{,X}}{K_{,X} + 2XK_{,XX}}$ models can be built such to have $C_s^2 \ll 1$ and $w \approx -1 \Rightarrow$ clustering DE

B) Coupled DE: Energy-momentum exchange can source perturbations in field with $w \approx -1$

C) Some modified gravity theories: extra degrees of freedom that cluster on large scales (mimicking DE clustering)

Coupled Dark energy

In general, scalar fields can couple to any other matter component

- Earlier we introduced the interaction in $\mathcal{L}$ example

$$S = \int \sqrt{-g} d\Omega [R + \mathcal{L}_\phi(\phi)] + S_m[\psi_m, e^{2Q\phi} g_{\mu\nu}] \rightarrow \text{Legendre transform} \rightarrow T^{\mu\nu}$$

- Alternatively, you can bring it directly in the conservation law

$$\boxed{\nabla_\mu T_{\text{Tot}}^{\mu\nu} = 0, \quad \nabla_\mu T_{(m)}^{\mu\nu} = J^\nu, \quad \nabla_\mu T_{(\phi)}^{\mu\nu} = -J^\nu, \quad J^i = 0 \text{ (homogeneity)} \quad i=1,2,3} \quad \text{energy-momentum transfer } m \leftrightarrow \phi$$

ϕ affects matter eq. of motion, i.e. it carries a force acting on matter: the fifth force

In general coupling of baryons can differ from the one of DM

e.g. $J_\phi = Q T_{(m)} \nabla_\mu \phi$, $Q = \text{coupling const.}$, $T_m = T^\mu{}_\mu$ trace of matter E-M tensor

$$\dot{\rho}_\phi + 3H(\rho_\phi c^2 + p_\phi) = -Q T_{(m)} \dot{\phi} \quad (1)$$

$$\dot{\rho}_m + 3H(\rho_m c^2 + p_m) = +Q T_{(m)} \dot{\phi} \quad (2)$$

Eq. of motion, additional source term

- New force is unscreened

$\Rightarrow$ extend to large distances

$\Rightarrow$ it appears as a modification of gravity $\nabla^2 \phi = -4\pi G f(a, r) \delta^2 \delta$ modified Poisson eq. $\tilde{G}(a, r)$

$\Rightarrow$ cosmic structure formation, is affected: $\delta_m'' + \frac{1}{2}(1-3w_{\text{eff}})\delta_m' - \frac{3}{2}f(a, r)\Omega_m \delta_m = 0$ growth eq.

constraints: large scales: cosmological (clustering, CMB, clusters, ...) + local = Solar system

$\hookrightarrow$ coupling must be weak or screening mechanism (Yukawa potential)

f : some function of time and space
 $\downarrow$

Scalar field with curvature coupling

$$\mathcal{L} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{2} \xi R \phi^2$$

$$(\Box - m^2 - \xi R) \phi = 0$$

$$\ddot{\phi} + 3H\dot{\phi} - \frac{1}{a^2} \nabla^2 \phi + \underbrace{(m^2 + \xi R)}_* \phi = 0$$

R = Ricci scalar (non minimal coupling)

ξ = dimensionless coupling constant $\phi \leftrightarrow R$

$\underbrace{g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi}$: curvature affects field only through derivatives (minimal coupling)
field propagates freely, following the geometry

$\tilde{m}^2 = m^2 + \xi R$: curvature affects field effective mass **and vice versa** (non-minimal)

" " field oscillations (even if $m=0$) $\nwarrow$ it affects the gravitational sector

Direct "interaction" $\phi \leftrightarrow R$

Motivations for the use of $\xi R \phi^2$

- 1) Natural choice from dimensional arguments : $[\xi]=0, [R]=2 \Rightarrow [\phi^2]=2$ only choice
- 2) $R \phi^2$ respects $\phi \rightarrow -\phi$ symmetry $\Rightarrow$ simpler theory
- 3) Covariant coupling : For $m=0$ the action is conformally invariant ($g \rightarrow \Omega^2 g$) for $\xi = 1/6$
 $\hookrightarrow$ Equivalent to some modified gravity theory
 Modifies $R \Rightarrow$ indirectly eq. of motion of matter (a 5th force : effective $\tilde{G}$ const)
- 4) Quantum corrections generates such term $\Rightarrow$ needed for renormalization
- 5) Cosmology : Alters inflationary dynamics (eg. Higgs inflation, ξ large)
 Changes particle production rates

Link to $f(R)$ modified gravity

$$\begin{array}{ll}
 S = \int \sqrt{-g} f(R) d\Omega + S_m(g_{\mu\nu}, \psi) + S(\tilde{g}_{\mu\nu}, \psi) & \text{Jordan frame} \quad \int \sqrt{-g} f(R) d\Omega \text{ modified gravity + standard } S_m, S_p \quad J^* = 0 \\
 \left| \begin{array}{l} \text{define field } \phi = \frac{dF(R)}{dR}, \quad \tilde{g}^{\mu\nu} = \phi g^{\mu\nu} \\ \text{Conformal transformation} \end{array} \right. & \updownarrow \\
 S' = \int \sqrt{-\tilde{g}} R d\Omega + S_m(A^2(\phi) \tilde{g}_{\mu\nu}, \psi) + S_p(\tilde{g}_{\mu\nu}, \psi) & \text{Einstein frame} \quad \int \tilde{R} \sqrt{-\tilde{g}} d\Omega GR + S_p + \text{modified } S_m \text{ (coupling)} \quad (J^* \neq 0)
 \end{array}$$

Most general scalar-tensor theory $(\phi, g_{\mu\nu})$

Hondenshi Lagrangian, See advance cosmology

$f(R) \subset$ Coupled DE $\subset$ Scalar-Tensor
 $Q = 1/6 \quad Q = \text{const} \quad \text{general}$

Appendix

• Example of constraint coming from type Ia supernova data

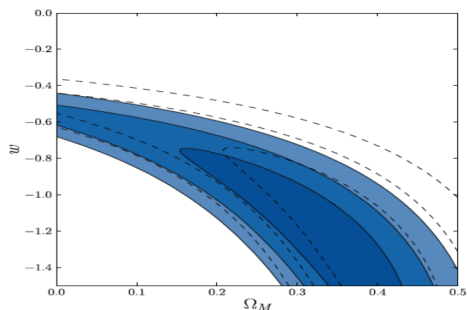


Figure 6.1.3: 68.3 %, 95.4 % and 99.7 % confidence level contours on $(\Omega_m^{(0)}, w_{DE})$ from the SN Ia observations (denoted as Ω_M and w in the figure) compiled in Amanullah et al. 2010ApJ...716..712A (@ AAS. Reproduced with permission).

Vacuum

Determines the background and physical properties (mass, symmetries, ...) of associated particle

Classical: $\phi_{vac} = \phi_0$ vacuum expectation value: field value at minimum of $V(\phi)$ $\left. \frac{dV}{d\phi} \right|_{\phi_0} = 0$

"rest position" of the field, with no disturbances, it remains there
classically there are no particles

QFT: fields are quantized

vacuum = lowest energy quantum state of the entire field (state with no particles)

$H|0\rangle = E_0|0\rangle$ $H = \text{Hamiltonian}$, $E_0 = \text{ground state energy}$

$a_p|0\rangle = 0$ state with no real particle, the annihilation "kill it"

$\langle 0|\phi^2|0\rangle \neq 0$ $\langle 0|\pi|0\rangle \neq 0$ but not truly empty $\pi = \dot{\phi}$ conjugate momentum

zero point fluctuations because of Heisenberg indet. princ.

virtual excitations - constant creation and annihilation
never becoming real particles

Real particle: on shell excitation with definite energy-momentum, detectable

Virtual p: off-shell, transient excitations contributing to field correlations

Semi classical gravity: vacuum energy and Λ

Extra

$$\boxed{G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G \langle T_{\mu\nu} \rangle_{\text{bare}}}$$

 $G_{\mu\nu} + \Lambda g_{\mu\nu}$: classical geometrical terms $\langle T_{\mu\nu} \rangle_{\text{bare}}$: QFT expectation value of vacuum and all speciesUsual Divergence issue in QFT

In flat space-time

$$\langle 0|T_{00}|0\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2} \hbar \omega_k \rightarrow \infty$$

mathematical artifact due to approximation:
continuum + point like interactions

In curved space-time it is worst

$$\langle T_{\mu\nu} \rangle_{\text{bare}} = \langle T_{\mu\nu} \rangle_{\text{finite}} + \langle T_{\mu\nu} \rangle_{\text{div}}$$

contains divergent terms arising from curvature

1) Regularize: eg. introduce Λ_{UV} cut-off ($\int_0^{\Lambda_{UV}} \dots$) or dimensional regularization

$$\langle T_{\mu\nu} \rangle_{\text{bare}} = (\text{finite state-dependent terms}) + \underbrace{\frac{\Lambda_{UV}^4}{16\pi^2} g_{\mu\nu}}_{A_0} + \underbrace{\frac{\Lambda_{UV}^2}{16\pi^2} G_{\mu\nu}}_{A_1} + \underbrace{\ln(\Lambda_{UV}) H_{\mu\nu}^{(2)}}_{A_2} + \dots$$

particles (excited states)

have covariant structure

individual parts might not be covariant $\Rightarrow$ (respecting general covariance)

$$\langle T_{\mu\nu} \rangle_{\text{div}}: A_i = \text{divergent const. } A_i \rightarrow \infty \text{ depending on regularization scheme and matter content}$$

scaling with powers of $\log(\Lambda_{UV})$

$$H_{\mu\nu}^{(2)} = \text{higher order "geometric" tensors based on curvature (e.g. } R^2, R_{\mu\nu} R^{\mu\nu}, \dots)$$

arising from quantum corrections

$$G_{\mu\nu}, H_{\mu\nu}^{(1)}, \dots \text{ local geometric terms coming from renormalization}$$

i.e. ultraviolet part (short distances) of quantum field couple to geometry

H: how gravitational field reacts to itself

$$\Lambda_0 g_{\mu\nu}: \text{is fully covariant}$$

" the only rank-2 tensor you can make from $g_{\mu\nu}$ alone with no derivatives" Lorentz invariant, isotropic, homogeneous energy-mom. tensor, same $\forall$ observer $\Rightarrow$ this is the vacuum energy term of the full $T_{\mu\nu}$ $A_0 = S_{\text{vac}}$ $\langle T_{\mu\nu} \rangle_{\text{vac}} = S_{\text{vac}} g_{\mu\nu}$ A_0 must be a constant because $\nabla_\mu T^{\mu\nu} = 0$ and $\nabla_\mu g^{\mu\nu} = 0$ 2) Renormalize

Absorb divergent parts into redefinitions of parameters (coupling strengths) in the gravitational sector

i.e. bring $\langle T_{\mu\nu} \rangle_{\text{div}}$ on the left side non physical divergent terms $\Lambda^{\text{bare}}, G^{\text{bare}}, H^{(2)}, \dots$

$$(1 + \bar{A}_1') G_{\mu\nu} + (\Lambda + \bar{A}_0') g_{\mu\nu} + \bar{A}_2' H_{\mu\nu}^{(2)} + \dots = \frac{8\pi G}{c^4} \langle T_{\mu\nu} \rangle_{\text{finite}} \quad \Big] \div (1 - \bar{A}_1')$$

$$\boxed{G_{\mu\nu} + \Lambda^{\text{ren}} g_{\mu\nu} + \alpha_2 H_{\mu\nu}^{(2)} + \dots = 8\pi G^{\text{ren}} \langle T_{\mu\nu} \rangle_{\text{finite}}}$$

renormalized geometric side

treat couplings ($\Lambda^{\text{ren}}, \alpha_i, G^{\text{ren}}$) as finite constants, they are what we measure! define through experiments

The geometric terms are modifications of gravity that arise naturally

Cosmological Inflation

Problems of the standard model

Friedmann model works very well but....

1) The horizon problem

- Universe is uniform on scales larger than the cosmological horizon
 - There are coherent structures larger than the horizon (CMB)
- } how can it be?!

2) The flatness problem:

- we see that today $\Omega_0 \sim 1$, why?!
- Friedmann model: $\Omega = 1$ ($k=0$) is very unstable
i.e. small deviation from $\Omega = 1$ rapid departure: $\rightarrow$ collapse $\Omega \rightarrow \infty$
fine-tuning $|\Omega_{\text{tot}} - 1| \ll 1$ $\rightarrow$ expansion $\Omega \rightarrow 0$

3) The relic problem:

- fundamental theories $\rightarrow$ e.g. monopoles... but they are not here nowadays, why?

4) The origin of cosmic structures:

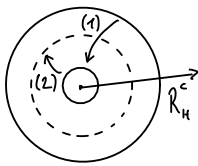
- Structure formation arise because of initial energy-density fluctuation
- Origin of these seeds?

One idea to fix everything: cosmic inflation

$\ddot{a} > 0$: phase of accelerated expansion of several e-folds at early times $t \ll t_{\text{eq}}$

- The horizon problem:

need phase in which the comoving horizon shrinks, i.e. $\dot{R}_H < 0$



- (0) R_H^c is "large", within R_H^c causal connection $\Rightarrow$ homogeneity, equilibrium
- (1) R_H^c shrinks $\Rightarrow$ scale of connected structures larger than R_H^c
- (2) R_H^c normal growth

$R_H^c \equiv \frac{c}{\dot{H}} = \frac{c}{\dot{a}}$ to shrink in time: $\frac{dR_H^c}{dt} = -\frac{c\ddot{a}}{\dot{a}^2} < 0 \Rightarrow \boxed{\ddot{a} > 0}$ accelerated expansion !!!

- The flatness problem:

$|\Omega_{t+1} - 1| = \frac{K_1 \dot{a}^2}{(\dot{a}^2 H^2)} \xrightarrow{\text{shrinks}} 0 \quad \text{Friedmann} \quad \text{shrinks} \quad \left. \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right| \Omega_{t+1} \rightarrow 1$

- The relic problem:

$S_{\text{relic}} \rightarrow 0$ because $a \uparrow$ (i.e. volume $\uparrow$)

- The origin of cosmic structures:

primordial quantum fluctuations δ_γ stretched and amplified

How much inflation do we need?

- # of e-folding needed to fix all problems $N \equiv \ln \frac{a_{\text{end}}}{a_{\text{start}}} \sim 60$ at least

$$|\Omega_{\text{tot}} - 1| = \frac{k^2 c^2}{a^2 H^2} \xrightarrow{\text{Friedmann}} 0$$

1) Curvature κ :

• Evolution of curvature $\Omega_\kappa = \frac{\kappa c^2}{a^2 H^2} = \frac{\kappa c^2}{a^2 H_0^2 (\Omega_{r0} a^{-4} + \Omega_{m0} a^{-3} + \Omega_{k0} a^{-2} + \Omega_\Lambda a^2)}$

radiation domination: $\Omega_\kappa \propto a^2$
 matter: $\Omega_\kappa \propto a$
 Λ : $\Omega_\kappa \propto a^2$

$\sim \Omega_{k0} \left(\frac{a_0}{a}\right)^2 \sim 0,01 \left(\frac{10^{-32}}{1}\right)^2 \approx 10^{-66}$ (radiation has the strongest dependency)

• Observations, today $|\Omega_{k0}| \lesssim 0,01$

• During inflation, exponential drop of Ω_κ : $a(t) \propto e^{Ht}$ $N \equiv \ln \frac{a_{\text{end}}}{a_{\text{start}}} \quad e^N = \frac{a_{\text{end}}}{a_{\text{start}}} \quad \Omega_\kappa \propto \frac{1}{a^2 H^2} = \Omega_{\kappa, \text{start}} e^{-2N} \stackrel{!}{\leq} 10^{-66}$

suppose $\Omega_{\kappa, \text{start}} = 1 \Rightarrow -2N \sim -66 \ln(10) \Rightarrow N \approx 75$

typical assumed value $N \geq 60$

2) Horizon: $R_H^c = \frac{a}{c} \cdot H$ need to shrink by a factor of 30 corresponding to approx 60 orders in a

3) Relics: The $N=60$ imposed by the other constraints, this is solved for free $\rho_m \propto a^{-3}$ given $N=60$, ρ_m drops by a factor of $10^{-3 \cdot 60} = 10^{-180}$

Which fluid provides $\ddot{a} > 0$?

$$\ddot{a} = -\frac{4\pi G}{3} \left(\rho + \frac{3p}{c^2} \right) > 0 \Rightarrow \rho + \frac{3p}{c^2} < 0 \Rightarrow p < -\frac{1}{3} \rho c^2 \Rightarrow \boxed{w < -\frac{1}{3}}$$

$p = w \rho c^2$
 $\downarrow$

- This fluid must dominate the expansion, i.e. all other species ($\Omega_m, \Omega_r, \dots$) are negligible, and get an exponential growth, eg. Λ dominated universe:
 from Friedmann equations it follows $a \propto \exp(\sqrt{\Omega_\Lambda} H_0 t)$ (de-Sitter model)

- It has to last long enough (60 e-foldings) to provide a sufficient expansion to fix all four problems

- It must end to become negligible to get the "normal" expansion. How?

Λ would be constant $\Rightarrow$ No $\Rightarrow$ need something else
 Dynamical scalar field: the inflation! Yes!

The simplest inflaton model

Scalar field ϕ (spin 0 particle)

• Lagrangian density

$$\mathcal{L} = -\frac{1}{2} c^2 \partial_\mu \phi \partial^\mu \phi - V(\phi)$$

• energy-momentum

$$T_{\mu\nu} = c^2 \partial_\mu \phi \partial_\nu \phi + \mathcal{L} g_{\mu\nu} = c^2 \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} c^2 \partial_\alpha \phi \partial^\alpha \phi g_{\mu\nu} - V(\phi) g_{\mu\nu}$$

Assumptions and model

Isotropy: only diagonal components of T are $\neq 0$

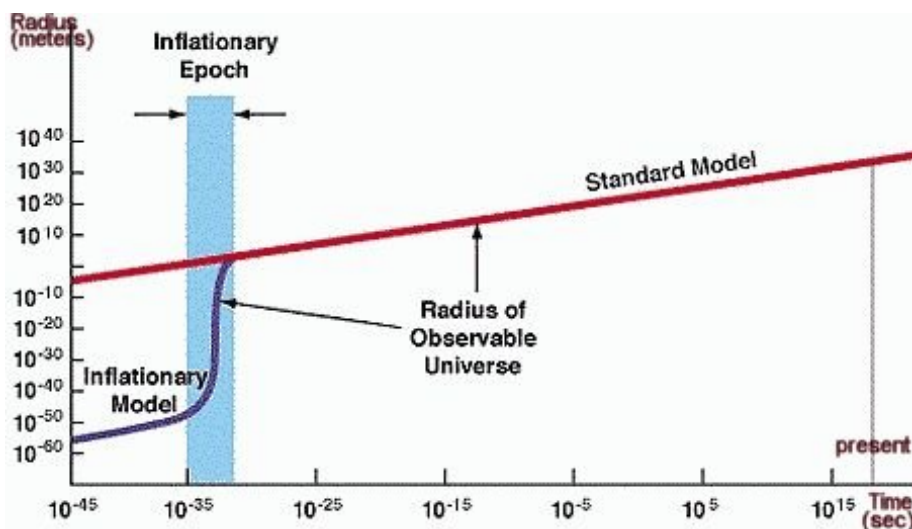
Homogeneity: $\delta_i \phi = 0$ * i.e. $\phi(t)$ only as well as isotropy (vector $\nabla \phi = 0$ = privileged direction)

$$\left\{ \begin{array}{l} T_{00} = \frac{1}{2} \dot{\phi}^2 + V(\phi) + \frac{1}{2} c^2 (\nabla \phi)^2 \stackrel{!}{=} \rho c^2 \quad \text{energy density} \\ T_{ii} = \frac{1}{2} \dot{\phi}^2 - V(\phi) - \frac{1}{6} c^2 (\nabla \phi)^2 \stackrel{!}{=} P_\phi \quad \text{pressure} \\ P = w \rho c^2 : \quad w = \frac{P}{\rho c^2} = \frac{\frac{1}{2} \dot{\phi}^2 - V}{\frac{1}{2} \dot{\phi}^2 + V} \quad \text{effective eq. of state} \Rightarrow w \text{ not constant in general!} \\ \ddot{\phi} + 3H\dot{\phi} + \frac{\delta V}{\delta \phi} = 0 \quad \text{eq. of motion "slow roll"} \quad \dot{\phi}^2 \ll V \\ H^2 = \frac{8\pi G}{3} \rho = \frac{8\pi G}{3} \left(\frac{1}{2} \dot{\phi}^2 + V \right) \quad \text{exponential growth in slow roll phase: } a(t) \propto e^{Ht}, \quad \rho \sim \text{const.} \end{array} \right. \quad \left. \begin{array}{l} \text{ideal fluid } T_{\mu\nu} = \left(\rho + \frac{P}{c^2} \right) u_\mu u_\nu + P g_{\mu\nu} \quad " \cdot " \equiv \frac{d}{c dt} \\ \text{conformal time} \end{array} \right\}$$

Same "set up" of dynamical dark energy but with additional constraints on duration and end.

Key difference: very high energy scale, $m_\phi c^2 \sim 6.5 \cdot 10^4 \text{ TeV}$ in contrast to DE (ultra light) !!!

↳ Different physics is possible here: rise in temperature and creation of particles (see reheating)



How do we need it to behave?

1) Accelerated expansion, i.e. $w_{\text{tot}} < -\frac{1}{3}$: $\rho < \rho_{\text{crit}} \Rightarrow \frac{1}{2}\dot{\phi}^2 - V(\phi) < -\frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \Rightarrow \boxed{\dot{\phi}^2 < V(\phi)}$

we can also obtain a constrain on $\ddot{\phi}$: $\boxed{2\ddot{\phi} < \frac{dV(\phi)}{d\phi}}$

$\frac{d}{dt}(\dot{\phi}^2) = 2\dot{\phi}\ddot{\phi} < \frac{dV}{d\phi}\dot{\phi}$

2) Exponential expansion, sufficient duration

(1) $\begin{cases} \dot{\phi} \ll V(\phi) \\ 2\ddot{\phi} < V'(\phi) \end{cases} \xrightarrow{!} \text{Slow roll condition}$

$\boxed{\dot{\phi} \ll V(\phi)} \quad (a)$ $\boxed{\ddot{\phi} \ll V'(\phi)} \quad (b) \quad V' \equiv dV/d\phi$ $\boxed{w = -1 + \epsilon \quad 0 < \epsilon \ll 1}$

- Ensures large number of e-foldings $N \equiv \log \frac{a_{\text{end}}}{a_{\text{init}}} \gg 1$ (~ 60) it must last long enough
- It simplifies eqs, from 2° to 1° order diff. eq. (but lose generality, need of an attractor behaviour)
- For some simple potential V , there are analytical solutions

3) Natural end of exponential expansion

when $\dot{\phi}$ and $\ddot{\phi}$ are not any more negligible: slow roll approximation breaks down
 ϕ starts to oscillate (damped oscillation) see eq. K-G eq. $w \rightarrow 0$ matter like

4) Production of standard particles (+DM)

by adding in the Lagrangian a coupling to matter $\mathcal{L}_{\text{int}}(\phi, \chi)$
 the inflaton energy goes in to creation of particles (reheating)

Apply approximation to the eq.s of motion

• Field: $\begin{cases} \rho_{\phi}^2 = \frac{1}{2}\dot{\phi}^2 + V \simeq V \Rightarrow \boxed{V(\phi) \sim \text{const.}} \text{ because } \dot{\phi} \ll V \\ p_{\phi} = \frac{1}{2}\dot{\phi}^2 - V \simeq -V \end{cases}$

• Dynamics $\begin{cases} H^2 = \frac{8\pi G}{3} \rho \simeq \frac{8\pi G}{3} V \Rightarrow \boxed{H^2 = \frac{8\pi G}{3} V(\phi)} \quad (1) \\ \ddot{\phi} + 3H\dot{\phi} + V' = 0 \Rightarrow \boxed{\dot{\phi} = -\frac{V'}{3H}} \quad (2) \end{cases}$

$\dot{\phi}^2 = \frac{V'^2}{9H^2} = \frac{V'^2}{9 \cdot \frac{8\pi G}{3} V} = \frac{1}{24\pi G} \frac{V'^2}{V} \ll V \Rightarrow \boxed{\epsilon(\phi) \equiv \frac{1}{24\pi G} \left(\frac{V'}{V}\right)^2 \ll 1}$

Additionally: $\ddot{\phi} = -\frac{d}{dt}\left(\frac{V'}{3H}\right) \ll \frac{V'}{2} \dots$ see appendix $\rightarrow \boxed{\eta(\phi) \equiv \frac{1}{24\pi G} \frac{V''}{V} \ll 1}$

Slow-Roll parameters

• Whenever $\epsilon \ll 1, \eta \ll 1$: slow roll is ensured $\Rightarrow$ we have inflation

• As soon as $\epsilon \sim 1, \eta \sim 1$: $\ddot{\phi}$ is "active" damped oscillations with damping factor $3H$
 $\Rightarrow$ Reheating, Inflation ends naturally

Example of potential

To get numbers out, we need an explicit potential, example:

free massive scalar field: $V(\phi) = \frac{1}{2}m^2\phi^2$, $V' = m^2\phi$ Chaotic inflation Linde (1983)
around global minimum at $\phi=0$ $V(\phi)$ approximated as a parabola

Time duration of inflation

Solve (2) for ϕ :

$$3H\dot{\phi} + V' = 0 \quad \dot{\phi} = -\frac{V'}{3H} = -\frac{V'}{3}\sqrt{\frac{3}{8\pi G V}} = -\frac{V'}{\sqrt{24\pi G V}} \quad \phi = \phi_i - \frac{V'}{\sqrt{24\pi G V}} t = \phi_i \left(1 - \frac{t}{\tau}\right) \quad \begin{array}{l} \phi_i: \text{initial value at } t=0 \\ \tau: \text{inflation time scale} \end{array}$$

First time field reaches minimum $\phi=0$ $\rightarrow$ all potential energy is converted in kinetic energy

$\Rightarrow$ slow roll conditions are broken $\dot{\phi} \neq 0$ and oscillations start

$\Rightarrow \phi=0 = \phi_i \left(1 - \frac{t}{\tau}\right) \quad t=\tau \Rightarrow$ yes, τ tells you duration of inflation

Inflation duration depends on initial conditions ϕ_i and field mass

$$\tau = \frac{\phi_i \sqrt{24\pi G V}}{V'} = \frac{\phi_i \sqrt{24\pi G \frac{1}{2}m^2\phi_i^2}}{m^2\phi_i} = \frac{\phi_i \sqrt{12\pi G}}{m} \quad \begin{array}{l} \text{depends on } \frac{\phi_i}{m}: \text{physical properties of inflaton!} \\ \text{constrained by number of required e-foldings } N \end{array}$$

Initial conditions to have sufficient inflation

• Number of e-foldings N : $N = \ln \frac{a}{a_i} = \int H dt \Rightarrow dN = H dt$ * exponential expansion

• Use eq. (2) and integrate in dN : $3H\dot{\phi} = -\frac{dV}{d\phi} \quad 3H\dot{\phi} = 3H \frac{d\phi}{dN} \frac{dN}{dt} = 3H^2 \frac{d\phi}{dN} = 3 \cdot \frac{8\pi G}{3} \frac{1}{2} m^2 \phi \frac{d\phi}{dN} = -m^2 \phi$
 $\Rightarrow \frac{d}{dN} \left(\frac{\phi^2}{2} - \frac{\phi_i^2}{2} \right) = -\frac{1}{N} \quad \Rightarrow \phi^2 = \phi_i^2 \exp\left[-\frac{2N}{N_i}\right]$

inflation ends when $\phi \rightarrow 0 \Rightarrow N = 2\pi G \phi_i^2 \Rightarrow \boxed{\phi_i^2 > \frac{N}{2\pi G}} \Rightarrow$ # of e-foldings N constrain ϕ_i !

- for minimum needed of $N \geq 60$: $\phi_i > 3m_P$ ϕ has units of a mass

- Assume initial condition values, ϕ_i , randomly distributed in space (this is why "chaotic inflation")
if a region meets this condition $\Rightarrow$ inflates as needed to produce our universe

- We can also generalize for a potential $V = \frac{\lambda}{2m} \phi^{2m} \Rightarrow \phi = \phi_i \exp\left[\frac{2\pi G}{m} (\phi_i^2 - \phi^2)\right]$, $\phi_i^2 > 60m/2\pi G$

- Can we reproduce in a lab. the conditions at time of inflation? No!

$m_\phi c^2 = 6.5 \cdot 10^4 \text{ TeV}$ max of LHC $\sim 13 \text{ TeV} \Rightarrow$ out of reach

What happens to all species already present before inflation?

- Clearly, during inflation, energy density of inflaton ρ_ϕ remains relevant despite strong volume increase
 in fact: $\begin{cases} V \text{ evolves very slowly (slow roll!)} & \text{i.e. } V \sim \text{const.} \\ \text{Energy density of the field} & \rho_\phi^2 = \frac{1}{2} \dot{\phi}^2 + V(\phi) \end{cases}$ $\dot{\phi} \ll V$ $V(\phi)$
 $\Rightarrow \rho_\phi \approx \text{const.}$
- But all other species are "diluted away": $\rho_r \propto a^{-4}$ $\rho_m \propto a^{-3}$ given $N=60$, ρ_i drops by a factor of $10^{-4 \cdot 60} = 10^{-240}$
 $10^{-3 \cdot 60} = 10^{-180}$
- $\Rightarrow$ Only the inflaton field is left but today we have radiation and matter where are they coming from?
- Also the temperature drops because of the expansion $T \downarrow$

Appendix

How does ϕ evolve? Eq. of motion

(1) Friedmann eq.: $H^2 = \frac{8\pi G}{3} \rho - \frac{Kc^2}{a^2} + \frac{\Lambda}{3} \rightarrow \begin{cases} H^2 = \frac{8\pi G}{3c^2} \left(\frac{1}{2} \dot{\phi}^2 + V(\phi) \right) \end{cases}$ $k=0$ very fast negligible at early times

(2) Continuity eq.: $\frac{d}{dt}(\dot{\phi}^2 c^2) + \rho \frac{d}{dt}(a^3) = 0 \rightarrow \begin{cases} \ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0 \end{cases}$
 $\uparrow$ acceleration $\uparrow$ friction $\uparrow$ force (noise)

$$\begin{aligned} \cdot 3\dot{\phi} \dot{\phi} c^2 + \dot{\phi} \dot{\phi} c^2 + \rho 3\dot{\phi} \dot{\phi} &= 0 \rightarrow 3H(\dot{\phi}^2 c^2 + \rho) + \dot{\phi}^2 c^2 = 0 \rightarrow \dot{\phi} = -\frac{3H\dot{\phi}^2}{c^2} \\ \cdot \dot{\phi} c^2 &= \frac{1}{2} \dot{\phi}^2 + V(\phi) \quad \dot{\phi} c^2 = \frac{1}{2} \dot{\phi}^2 + V(\phi) \dot{\phi} \quad -3H\dot{\phi} = \dot{\phi} \ddot{\phi} + V'(\phi) \dot{\phi} \quad \checkmark \end{aligned}$$

Slow-roll parameters

Dynamics $\begin{cases} H^2 = \frac{8\pi G}{3} \rho - \frac{Kc^2}{a^2} + \frac{\Lambda}{3} \approx \frac{8\pi G}{3} \left(\frac{1}{2} \dot{\phi}^2 + V(\phi) \right) \quad H^2 = \frac{8\pi G}{3} V(\phi) \quad (1) \\ \ddot{\phi} + 3H\dot{\phi} + V' = 0 \quad \dot{\phi} = -\frac{V'}{3H} \quad \dot{\phi}^2 = \frac{V'^2}{9H^2} = \frac{V'^2}{9 \frac{8\pi G}{3} V} = \frac{1}{24\pi G} \frac{V'^2}{V} \ll V \Rightarrow \epsilon(\phi) \equiv \frac{1}{24\pi G} \left(\frac{V'}{V} \right)^2 \ll 1 \end{cases}$

Additionally: $\dot{\phi} = -\frac{V'}{3H} \quad \ddot{\phi} = -\frac{d}{dt} \left(\frac{V'}{3H} \right) = -\frac{V''\dot{\phi}}{3H} + \frac{V'H}{3H^2} = -\frac{V''\dot{\phi}}{3H} + \frac{V'\dot{\phi}}{6VH} \ll \frac{V'}{2} = -\frac{3H\dot{\phi}}{2} \quad \downarrow \div H\dot{\phi}$

$$\frac{d\epsilon}{dt} = \frac{1}{24\pi G} \frac{V''^2}{V} - \frac{V''}{24\pi G} \frac{V'}{V} \quad \left(\frac{\dot{H}}{H} = \frac{4\pi G}{3c^2 H^2} V' \dot{\phi} = \frac{\dot{\phi}}{2} \frac{V'}{V} \right)$$

$$\Rightarrow \frac{V''}{3H^2} - \frac{V'^2}{6VH^2} = \frac{1}{8\pi G} \left(\frac{V''}{3V} - \frac{V'^2}{6V^2} \right) \equiv 3\eta(\phi) - \frac{3}{2}\epsilon(\phi) \ll \frac{3}{2} \sim 1 \quad \eta(\phi) \equiv \frac{1}{24\pi G} \frac{V''}{V} \ll 1$$

Why is the inflaton field called a 'condensate'?

A spatially homogeneous classical field $\sim$ coherent state containing an extremely large number of quanta in the same mode

Not just an analogy:

$$|\alpha\rangle$$

coherent state

$$\hat{S}_n |\alpha\rangle = \alpha |\alpha\rangle$$

all quanta in the same mode $(k=0, \text{the homogeneous mode})$

$$N = |\alpha|^2 \gg 1$$

average number (not finite particle number)

$$\langle \alpha | \phi | \alpha \rangle = \phi_{\text{classical}}(t)$$

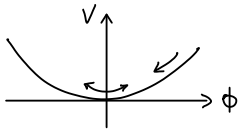
classical field = expectation value of a quantum operator

coherent state = macroscopic occupation of one mode

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0$$

analogous to a Bose-Einstein condensate oscillating in an harmonic trap

$\checkmark$ recall: slow roll $\approx 1 \sim$ vacuum

End of inflation, reheating and particle creationSlow roll but eventually $\varepsilon \approx 1, \eta \approx 1$ 

end

 $\Rightarrow \dot{\phi}$ and $\ddot{\phi}$ become relevant $\Rightarrow \ddot{\phi} + 3H\dot{\phi} + V' = 0$ $\Rightarrow$ Oscillations: matter expansion $\Rightarrow H = \frac{2}{3t} \quad \rho \propto t^{-3}$ $\Rightarrow$ damped oscillations of ϕ ($3H\dot{\phi}$ friction): natural end
 $3H\dot{\phi}$ friction does NOT produce particles! Just GR non-conservation+ include coupling to matter to let ϕ decay into particles *
* ϕ stores kinetic energy $\dot{\phi}^2$ Boltzmann eq. $\ddot{\phi} + 3H\dot{\phi} + T_\phi \dot{\phi} + V'(\phi) = 0$ effective friction T_ϕ , $3H\dot{\phi}$ does not produce particles!A deeper view $\mathcal{L} = \mathcal{L}_\phi(\phi) + \mathcal{L}_\chi(\chi) - \mathcal{L}_{int}(\phi, \chi) \quad \nabla_\mu T_\phi^{\mu\nu} = 0: \quad \nabla_\mu T_\phi^{\mu\nu} = -J^\nu, \quad \nabla_\mu T_\chi^{\mu\nu} = J^\nu \quad J^\nu = T_\phi \dot{\phi} \delta^\nu_0$ (simplest) ρ_ϕ energy transfer ρ_χ interaction eg. $\mathcal{L}_{int} = -g\phi\chi^2$ decay $\phi \rightarrow \chi + \chi \quad m_\phi > 2m_\chi \quad \chi$ light scalareg. $\mathcal{L}_{int} = -g\phi\bar{\psi}\psi$

"

 $\phi \rightarrow \bar{\psi} + \psi \quad \psi$ fermions(Yukawa) eg. $\mathcal{L}_{int} = -\frac{\phi}{4\Lambda} F_{\mu\nu} F^{\mu\nu}$

"

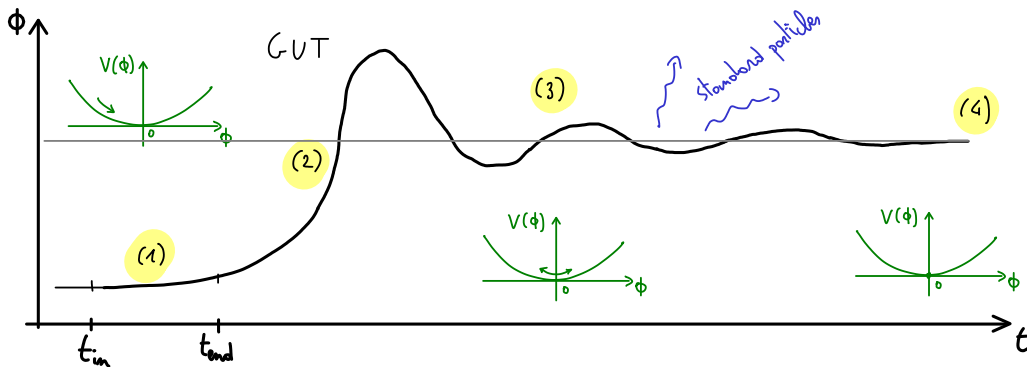
gauge bosons via dimension-5 operator

(Planck suppression)

eg. $T_\phi \sim f \frac{m_\phi^2}{M_{Pl}^2}$

(coupling to gravity only)

from dimensional arguments

 $\Rightarrow$ Kinetic energy is transformed into standard particles (+ dark sector + DM?)Reheating $T \uparrow \quad T_{reh} \sim 10^9 - 10^{16} \text{ GeV}$: Hot Big Bangknown particles have rest mass $m \ll 10^{15} \text{ GeV} \Rightarrow$ relativistic particles Thermalized
radiation domination era needed for observed Big Bang Nucleosynthesis (BBN)

stage	cosmic epoch	dominant component	dominant term	w	relevant	reactions
slow-roll	inflation	inflaton ϕ	$V(\phi)$	-1	$\rho_\phi = \text{const}, \rho_i \downarrow, T \downarrow$	$H \gg T_\phi$
fast-roll	stiff-matter	"	$\frac{1}{2} m \dot{\phi}^2$	1		"
{ oscillatory decay }	{ matter-like reheating }	"	{ $\dot{\phi}^2 \sim V$ transition }	0	$\rho_\phi \propto t^{-3} \quad \rho_i = 0 \quad T \downarrow$	$H \gg T_\phi$
		" $\rightarrow$ particles		0	$\rho_\phi \propto t^{-3} \quad \rho_i \uparrow \quad T \uparrow$	$H \sim T_\phi$
Thermalization	radiation-like	relativistic particles	$\gamma, e^+, e^-, \eta, m, \chi$	1/3	Hot Big-Bang	$H \ll T_\phi$

End of inflation: details

1) Inflation phase

- $H \gg T_\phi$! it does not dilute ϕ
- $\epsilon \ll 1, \eta \ll 1$: slow roll $\Rightarrow V(\phi)$ dominant $\Rightarrow w \simeq -1$ accelerated expansion, $\rho_\phi \simeq \text{const.}$, $m \ll H$
- $\epsilon(\phi_{\text{end}}) \simeq 1$: inflation ends $\ddot{\epsilon}$ changes sign
- Next steps are not separate, continuous transitions

2) Transition phase "Fast roll"

- Inflation is still the relevant specie, all the rest has been diluted away
- $\dot{\phi}$ grows: $\dot{\phi} \propto V$ or even larger $\Rightarrow w = 1$ for a short period (few Hubble times H^{-1})
little expansion but sets conditions for following phase

3) Oscillations

- After reaching the ϕ minimum: coherent oscillations (condensate of nearly zero momentum scalar particles)
 $\frac{1}{2} \dot{\phi}^2 \propto V$, $\langle w \rangle = 0$ matter like cosmic expansion $\propto t^{2/3}$ $\rho \propto a^{-3}$
- Around minimum: $V(\phi) \simeq \frac{1}{2} m^2 \phi^2$ (Taylor expansion), or for simplest model
- $\dot{\phi}$ not negligible: $3H\dot{\phi}$ "Hubble friction" is dumping the oscillations
- Ends when inflaton decays efficiently into other particles (for decay you need a coupling with matter)

3) Reheating

dumped oscillations $\Rightarrow$ kinetic energy is transferred to some thing else because of "friction"
i.e. inflaton decays into whatever it couples to (coupling = energy transfer)
 $\hookrightarrow$ standard model particles or other fields and than standard particles

Preheating:

- non perturbative, parametric resonance from some oscillating modes (Mathieu/Hill eq.)
- Not all models have preheating: Highly model dependent on chosen coupling
eg. $\mathcal{L}_{\text{int}} = -\frac{1}{2} \phi^2 \chi^2$ self annihilation $\phi + \phi \rightarrow \chi + \chi$, T_ϕ not sufficient, you need an effective decay rate
- still matter like expansion
- short explosive reheating, Very rapid non thermal equilibrium
- strong energy transfer of inflaton to "something else" in few oscillations
- Can produce large fluctuations and need non-linear dynamics ($\delta_i \phi \neq 0$, gravitational waves, ...)
- Ends when resonance subides

Reheating: Oscillations $\Rightarrow \langle \rho \rangle = 0, \langle \rho_r \rangle \propto a^{-3}, \rho_\phi \neq 0$ *matter like*
 efficient (perturbative) decay when decay rate $\Gamma_\phi \approx H(t)$
 here, "standard" particles creation: Reheating (+ hidden sector: dark radiation, sterile neutrinos, ...)
 condensate description replaced by multi-field/particle description
 Ends when universe is radiation dominated and ρ_ϕ is gone

• Note: $3H\dot{\phi}$ is a friction term, kinetic energy get dissipated. Where does it go?
 Not in particles production!
 Global energy is not conserved in General Relativity! : $\dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = 0$ is $\nabla_\mu T^{\mu\nu} = 0$
 the field work against the expansion : $\dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = 0$ is $dE + p dV = 0$
 $\Rightarrow$ Need coupling $\sim$ effective friction term Γ_ϕ

• Coupling with matter e.g. $\xi^2 \phi^2 \chi^2$ (annihilation $\phi + \phi \rightarrow \chi + \chi$) $\Rightarrow$ decay rate Γ_ϕ = effective friction term
 $\int \phi \chi^2$ (decay $\phi \rightarrow \chi + \chi$ for $m_\phi > 2m_\chi$)

$\nabla_\mu T^{\mu\nu} = 0$: $\nabla_\mu T^{\mu 0} = -J^0$, $\nabla_\mu T^{\mu i} = J^i$ $v=0$: set $J^0 = \Gamma_\phi \rho_\phi$ (simplest) the more ρ_ϕ , the more energy can be transferred
 Γ_ϕ (decay) rate at which energy is transferred
 see appendix...

• Boltzmann eq.

$\ddot{\phi} + 3H\dot{\phi} + \Gamma_\phi \dot{\phi} + V'(\phi) = 0$ $\downarrow \cdot \dot{\phi}$ Γ_ϕ : phenomenological decay/friction, models energy loss to produce particles

$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) + \Gamma_\phi \dot{\phi} = 0$ $\downarrow \dot{\phi} \ddot{\phi} = \frac{1}{2} \frac{d}{dt}(\dot{\phi}^2)$ $V'(\phi) \dot{\phi} = \frac{dV(\phi)}{dt}$

$\frac{d}{dt} \left(\frac{\dot{\phi}^2}{2} + V(\phi) \right) + 3H\dot{\phi}^2 + \Gamma_\phi \dot{\phi}^2 = 0$ $\downarrow m_\phi \gg H$ (rapid oscillations) $V = \frac{1}{2} m^2 \phi^2 \Rightarrow \langle \dot{\phi}^2 \rangle = \langle \rho_\phi \rangle$ ★

$\Rightarrow \dot{\rho}_\phi + 3H\langle \rho_\phi \rangle = -\Gamma_\phi \langle \rho_\phi \rangle$ $\langle - \rangle$ time average $\downarrow m_\phi \approx \frac{\rho_\phi}{m_\phi}$ (average over many oscillations is valid)
 $\dot{n}_\phi + 3Hn_\phi = -\Gamma_\phi n_\phi$ Number of particles $\downarrow \Gamma_\phi \dot{\phi}$ valid when decay products thermalize
 $\langle \dot{\phi}^2 \rangle \approx \langle \rho_\phi \rangle$ valid for quadratic potential

$m \gg H$ $\ddot{\phi} + m_\phi^2 \phi \approx 0$ $\downarrow \cdot \phi$ and $\langle - \rangle$ time average $\overline{\phi \dot{\phi}} = 0$ $\overline{\dot{\phi}^2} = \frac{1}{T} \int_0^T \dot{\phi}^2 dt = \frac{1}{T} \int_0^T \phi \frac{d\dot{\phi}}{dt} dt = \frac{1}{T} \left(\phi \dot{\phi} \Big|_0^T - \int_0^T \phi^2 dt \right) = \frac{1}{2} \langle \dot{\phi}^2 \rangle$
 $-\langle \dot{\phi}^2 \rangle + m^2 \langle \phi^2 \rangle = 0 \Rightarrow \langle \dot{\phi}^2 \rangle = m^2 \langle \phi^2 \rangle$ $\langle \rho_\phi \rangle = \frac{1}{2} \langle \dot{\phi}^2 \rangle + \frac{1}{2} m^2 \langle \phi^2 \rangle = m^2 \langle \phi^2 \rangle$ ★

Energy exchange, particles production

$\begin{cases} \dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = -\Gamma_\phi \rho_\phi & \begin{matrix} \rho_\phi \\ \rho_r \end{matrix} & 3H\rho_\phi: \text{dilution of inflaton, } -\Gamma_\phi \rho_\phi: \text{decay into radiation} \\ \dot{\rho}_r + 3H(\rho_r + p_r) = +\Gamma_\phi \rho_\phi & & 4H\rho_r: \text{reheating of radiation, } +\Gamma_\phi \rho_\phi: \text{build up of radiation } \sim \text{relativistic fluid} \\ H^2 = \frac{8\pi G}{3}(\rho_\phi + \rho_r) & \uparrow p_r = \frac{1}{3}\rho_r & p_\phi = 0 \end{cases}$

Solution for $U(\phi) = \frac{1}{2} m^2 \phi^2$ $p_\phi = 0$ $\rho_\phi \propto \dot{\phi}^2$ (if no decay)

$$\begin{cases} (1) \dot{\rho}_\phi + 3H\rho_\phi = -\Gamma_\phi \rho_\phi \\ (2) \dot{\rho}_R + 4H\rho_R = +\Gamma_\phi \rho_\phi \\ (3) H^2 = \frac{8\pi G}{3} (\rho_\phi + \rho_R) \end{cases}$$

• Exponential dumping of ρ_ϕ due to particles creation (early phase)

$H \ll T$ reactions not yet efficient

$\rho_\phi \gg \rho_R$: $\rho_\phi = \frac{3H^2}{8\pi G}$, oscillations (matter domination): $w=0$, $H = \frac{2}{3}t^{-1}$, $a = a_i t^{2/3}$

$$(1) \dot{\rho}_\phi + (2t^{-1} + \Gamma_\phi) \rho_\phi = 0 \quad \frac{d\rho_\phi}{\rho_\phi} = -(2t^{-1} + \Gamma_\phi) dt \quad \ln \rho_\phi \Big|_{t_i}^t = -2 \ln t \Big|_{t_i}^t - \Gamma_\phi t \Big|_{t_i}^t \quad \rho_\phi(t) = \rho_{\phi,i} \left(\frac{t}{t_i} \right)^{-2} e^{-\Gamma_\phi(t-t_i)}$$

$$t = \left(\frac{\rho}{\rho_i} \right)^{3/2} \quad \boxed{\rho_\phi(t) = \rho_{\phi,i} \left(\frac{a}{a_i} \right)^{-3} e^{-\Gamma_\phi(t-t_i)}} \quad t_i = 0$$

• Time of reheating

$t_{\text{reh}} = \frac{2}{3} H^{-1} \simeq \frac{2}{3} \Gamma_\phi^{-1}$ Matter domination ($H = \frac{2}{3}t^{-1}$) and efficient reactions ($H \sim \Gamma_\phi$)

• Reheating temperature of radiation domination (late phase)

Reheating time: radiation domination ($\rho_\phi \ll \rho_R$) when reactions are efficient ($H \sim \Gamma_\phi$)

Friedmann eq.: $H^2 = \frac{8\pi G}{3} (\rho_\phi + \rho_R) \rightarrow$ Density $\rho_R(t_{\text{reh}}) = \frac{3H^2}{8\pi G} \simeq \frac{3\Gamma_\phi^2}{8\pi G}$

Thermal bath: $\rho_R = \frac{\pi^2}{30} g_* \left(\frac{k_B T}{\hbar c} \right)^4$ relativistic plasma $\Rightarrow (k_B T)^4 = \frac{g_*}{g_*} \left(\frac{\hbar c}{\pi^2} \right)^3 30 = \frac{3\Gamma_\phi^2}{8\pi G} \left(\frac{\hbar c}{\pi^2} \right)^3 30 = \frac{90\Gamma_\phi^2}{8Gg_*} \left(\frac{\hbar c}{\pi} \right)^3$

$g_* \sim 100$ effective number of relativistic degrees of freedom

From CMB observations (tensor-to-scalar ratio bound) $T_{\text{reh}} < 10^{15} \text{ GeV}$

• Ultra-relativistic products, the Hot Big Bang

During inflation $T \downarrow$ decrease (because of dramatic expansion, adiabatic)

Reheating, energy release in medium $\Rightarrow T \uparrow$, T_{max} settles at $T_{\text{reh}} \sim 10^9 - 10^{15} \text{ GeV}$ (depends on model, Γ_ϕ , $U(\phi)$)

Hot Big Bang

known particles have rest mass $m \ll 10^{15} \text{ GeV}$

decay products are ultra-relativistic (rest energy $\ll$ kinetic energy): expansion is "radiation" dominated $w = 1/3$

photons, gluons, relat. e^- and e^+ , neutrons n , and any other light species (Dark matter?)

Radiation domination epoch $\Rightarrow T \downarrow$ again because adiabatic expansion

Neutrinos (small mass) remain ultra-relativistic till today

Dark-Matter and other massive particles became non-relativistic earlier (Cold Dark Matter)

4) Thermalization

Rapid interactions (through gauge and Yukawa couplings) $\rightarrow$ thermalization $\rightarrow$ effective

The new particles interact $\rightarrow$ scattering $\rightarrow$ local thermal equilibrium $\rightarrow$ nearly perfect black-body spectrum

$w = 1/3$ $\rho \propto a^{-4}$ Standard cosmological model, radiation dominated

why?

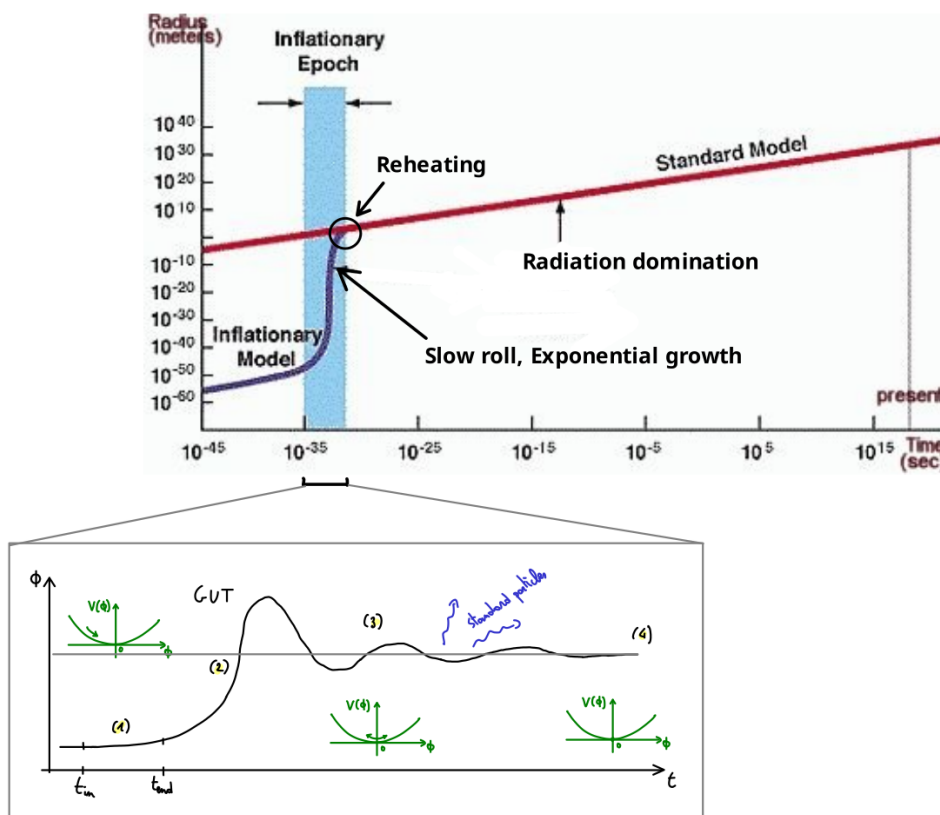
Radiation domination is needed:

- 1) $w = 1/3 \Rightarrow a(t) \propto t^{1/2}$ matching expected expansion rate for successful BBN and CMB
- 2) BBN requires well defined γ , e^- , n
- 3) Observed entropy and photons/baryons ratio today needs this thermal phase
if universe expansion matter dominated for too long (oscillation phase) $\Rightarrow$ BBN failure

5) Afterward

$T \downarrow \Rightarrow$ some become non relativistic (like quarks, gluons) $\rightarrow$ form matter

Cold = non relativistic: Cold Dark Matter (CDM)



Appendix

More on interaction rate and coupling strength

e.g. in Lagrangian: $\mathcal{L}_{int} = -\bar{\psi} \phi \psi$ $\phi \rightarrow \psi + \bar{\psi}$ with $m_\phi > 2m_\psi$ decay
 $\mathcal{L} = -\frac{1}{2} \dot{\phi}^2 - \frac{1}{2} m_\phi^2 \phi^2$ $\phi + \phi \rightarrow \chi + \chi$ scattering perturbative annihilation $\Gamma_{ann} \sim \frac{\xi^4 \phi^2}{512 \pi m_\phi}$

$$\mathcal{L} = \mathcal{L}_\phi(\phi) + \mathcal{L}_\chi(\chi) - \frac{1}{2} \xi^2 \phi^2 \chi^2 : \sum_\mu \frac{\delta \mathcal{L}}{\delta \phi_{,\mu}} - \frac{\delta \mathcal{L}}{\delta \phi} = 0, \quad \Box \phi + V'(\phi) - \frac{\delta \mathcal{L}_{int}}{\delta \phi} = 0, \quad \frac{\delta \mathcal{L}_{int}}{\delta \phi} = -\xi^2 \phi \chi^2, \quad \underbrace{\Box \phi + V'(\phi)}_{=0} + \xi^2 \phi \chi^2 = 0$$

$$T_{\mu\nu} = T_{(\phi)}^{\mu\nu} + T_{(\chi)}^{\mu\nu} + T_{(int)}^{\mu\nu} : T_{\phi}^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - g^{\mu\nu} \mathcal{L}_\phi \quad \sum_\mu T_{(\phi)}^{\mu\nu} = \underbrace{(\Box \phi + V'(\phi)) \partial^\nu \phi}_{=0} = -(\xi^2 \phi \chi^2) \partial^\nu \phi \quad \text{as a source term}$$

$$\begin{aligned} v=0: \dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) &= -S(t) & S(t) &= \xi^2 \langle \phi \phi \chi^2 \rangle \quad \text{energy loss rate} \\ &= 2 m_\phi (m_\phi m_\phi \langle \sigma v \rangle_{\phi\phi \rightarrow \chi\chi}) \quad \text{as loss of energy density} \\ &\quad \uparrow \quad \uparrow \\ &\quad \rho_\phi \approx m_\phi m_\phi \quad (\text{non relativistic}) \\ &\quad 2 \text{ particles of } \phi \text{ annihilate} \\ &= \Gamma_{ann} \rho_\phi \quad \Gamma_{ann} = m_\phi \langle \sigma v \rangle \end{aligned}$$

$$\rho_\phi = \frac{1}{2} m_\phi^2 \phi^2 \quad (\text{quadratic potential near minimum})$$

$$\begin{aligned} M_\phi &= \rho_\phi / m_\phi = \frac{1}{2} m_\phi \phi^2 & \sigma v &\sim \frac{|M|^2}{16\pi s} & M &= -i\xi^2 \quad (\text{tree amplitude for } \phi\phi \rightarrow \chi\chi) \\ \Gamma &\approx m_\phi \sigma v \sim \frac{1}{2} m_\phi \phi^2 \cdot \frac{\xi^4}{256\pi m_\phi^2} & & & & \text{symmetric scalar} \end{aligned}$$

$$\text{Same for the matter field: } \sum_\mu \frac{\delta \mathcal{L}}{\delta \chi_{,\mu}} - \frac{\delta \mathcal{L}}{\delta \chi} = 0 \Rightarrow \dot{\rho}_\chi + 3H(\rho_\chi + p_\chi) = +S(t)$$

Effective number of relativistic degrees of freedom g_*

• order of magnitude estimate (instantaneous value), should include "history" with also preheating

$$g_* = \sum_{\text{bosons}} g_i + \frac{7}{8} \sum_{\text{fermions}} g_i = 106.75 \quad g_i = \# \text{ internal degrees of freedom (spin, color, ...)}$$

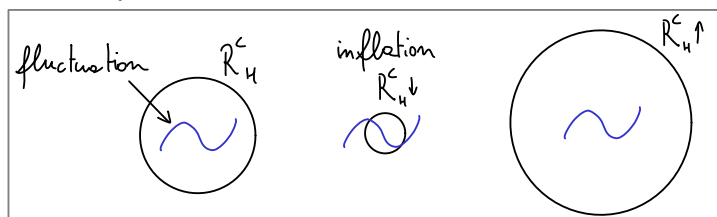
↑
difference between F-D and B-E statistics (Fermi-Dirac weight)

$$\begin{aligned} \text{Bosons:} & \left. \begin{array}{ll} \text{gluons (g)} & 8_{\text{colors}} \times 2 \text{ spins} = 16 \\ W^\pm, Z, \gamma & 4 \times 2 = 6 \\ \text{Higgs doublet} & 4 \end{array} \right\} g_b = 16 + 6 + 4 = 28 \\ \text{Fermions:} & \left. \begin{array}{ll} \text{Quarks (u, d, c, s, t, b)} & 72 \\ \text{Leptons (e, } \mu, \tau, \nu_e, \nu_\mu, \nu_\tau) & 18 \end{array} \right\} g_f = 90 \end{aligned} \quad \left. \begin{array}{l} g_b = 28 \\ g_f = 90 \end{array} \right\} g_* = 28 + \frac{7}{8} 90 = 106.75$$

The seeds of cosmic structure formation

Bottom line:

- Cosmic structures grow from initial primordial (Bunch-Davies) vacuum quantum fluctuations at $t \approx t_{\text{Planck}}$
 $\langle 0 | \phi_k | 0 \rangle$ expectation value of the ground level of the inflaton field at wave number k
 Creation/Annihilation of virtual particles $\leftarrow \dots \rightarrow$
- During inflation: R_H^C shrinks and pairs (fluctuations) are no more in causal contact $\Rightarrow$ "freeze in"
 they freeze when they are larger than the horizon which shrinks because of inflation



- Being a random process, all scales, k , are equally affected

Quantum fluctuations can in general excite all 3 types

There are 3 types of metric perturbations

Metric decomposition: Scalar-Vector-Tensor decomposition (SVT)

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + \delta^2 g_{\mu\nu}^{(1)} \quad |g_{\mu\nu}^{(1)}| \ll |g_{\mu\nu}^{(0)}| \quad g^{(1)} = \begin{pmatrix} -2\psi & w_i \\ w_i & 2\phi\delta_{ij} + h_{ij} \end{pmatrix} \quad \text{most generic perturbed metric}$$

$$ds^2 = \delta^2(t) \left\{ -(1+2\psi)c^2 d\tau^2 + 2w_i c d\tau dx^i + [(1+2\phi)\delta_{ij} + h_{ij}] dx^i dx^j \right\} \quad \text{Flat and in Newton gauge}$$

$$\left. \begin{array}{lll} \psi(x^\nu), \phi(x^\nu) & \text{two scalar fields} & 2 \text{ d.o.f.} \\ w_i(x^\nu) & \text{3-vector fields} & 3 \text{ d.o.f.} \\ h_{ij}(x^\nu) & \text{3-tensor, traceless, symmetric} & -1+6=5 \text{ d.o.f.} \end{array} \right\} 10 \text{ degrees of freedom}$$

- 1) vector ones decay rapidly $\Rightarrow$ neglect
- 2) scalar ones, difficult derivation: need to perturb simultaneously metric and scalar field
- 3) tensorial ones: vacuum solution of linearized Einstein eq.s $\rightarrow$ harmonic oscillations
 $\Rightarrow$ straightforward quantization, easier derivation

Predicted power spectrum

A qualitative but complete approach for scalar fluctuations:

- solve eq. for field amplitude $\{H^2, \dots\}$
 r.m.s. of fluctuations: $\sqrt{\langle \delta\varphi^2 \rangle} \propto \frac{H^2}{\dot{\varphi}} \sim \text{const}$ because of slow roll ($\delta\varphi \sim \text{metric fluctuations}$)
 $\rho \sim \text{const}$

- Fluctuations of the grav. potential caused by a spherical overdensity:

$$\delta\varphi = \frac{G\delta M}{R} = \frac{G}{R} \delta\rho \frac{4\pi}{3} R^3 = \frac{4\pi G}{3} \frac{\delta\rho}{\dot{\varphi}^2} (\dot{\varphi} L)^2 \propto L^2 = \text{const.}$$

$R = \dot{\varphi} L$ physical radius

$\Rightarrow \delta\varphi$ remains const. during inflation

Friedmann eq.: $\otimes$

$$H^2(\dot{\varphi}) = \frac{8\pi G}{3} \rho - \frac{K}{a^2} \quad \rho = \frac{3H^2}{8\pi G} + \frac{3K}{8\pi G a^2} = \rho_m + \frac{3K}{8\pi G a^2} = \frac{\rho}{\Omega} + \frac{3K}{8\pi G \Omega^2} \quad \rho \dot{\varphi}^2 \left(1 - \frac{1}{\Omega}\right) = \frac{3K}{8\pi G} = \text{const.}$$

Fluctuations: $\rho \dot{\varphi}^2 \left(\frac{\Omega-1}{\Omega}\right) = \rho \dot{\varphi}^2 \left(\frac{1+\delta\Omega-1}{\Omega}\right) \approx \rho \dot{\varphi}^2 \delta\Omega = \dot{\varphi}^2 \delta\rho = \text{const.} \Rightarrow \boxed{\delta\rho \propto \dot{\varphi}^{-2}}$

For a linear overdensity $\Omega = 1 + \delta\Omega \quad \delta\Omega \ll 1 \quad \delta\rho = \rho \delta\Omega$

- $R = \dot{\varphi} L$ changes by ~ 30 orders of magnitude during inflation $\Rightarrow$ all physical scales are involved (also macroscopic ones)

- Fluctuations power spectrum:

Power spectrum: $\langle f^*(\vec{k}) f(\vec{k}') \rangle = (2\pi)^3 P(k) \delta(\vec{k}-\vec{k}')$ here $f = \delta\varphi$
 provides a full description of Gaussian random fields

• Poisson eq. to link $\delta\varphi$ to $\delta\rho$: $\nabla^2 \delta\varphi = 4\pi G \delta\rho$ Fourier space $-k^2 \hat{\delta\varphi}(k) \propto \hat{\delta\rho}(k)$

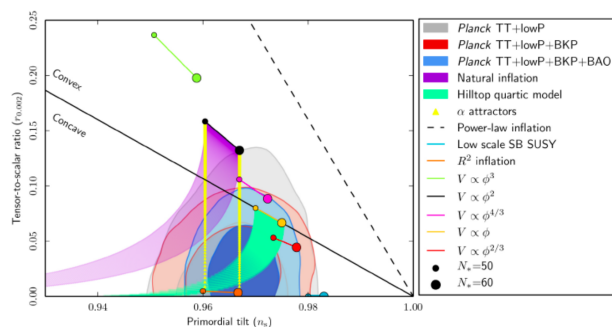
• fluctuations power spectrum $k^3 P_i(k) \propto |\hat{\delta\varphi}|^2 \propto k^4 |\hat{\delta\varphi}|^2 \sim k^4 \text{const.}$

$\Rightarrow P_i(k) \propto k$ Harrison-Zel'dovich-Peebles spectrum

Precise computations $P_i(k) \propto k^{n_s} \quad n_s \approx 0.96$ (!) n_s : scalar spectral index observable

- We have superposition of a very large number of independent vacuum fluctuations (central limit theorem) $\Rightarrow$ expected to be Gaussian random field (!) observable

Deviations from that are called non-Gaussianities



Planck 2015 (xx) Abol et al

THE EARLY UNIVERSE

by GUILLEM DOMENECH

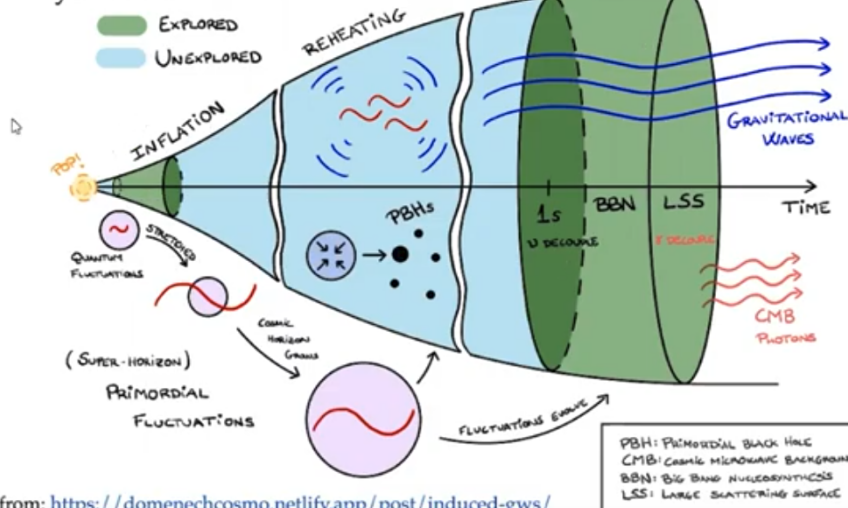
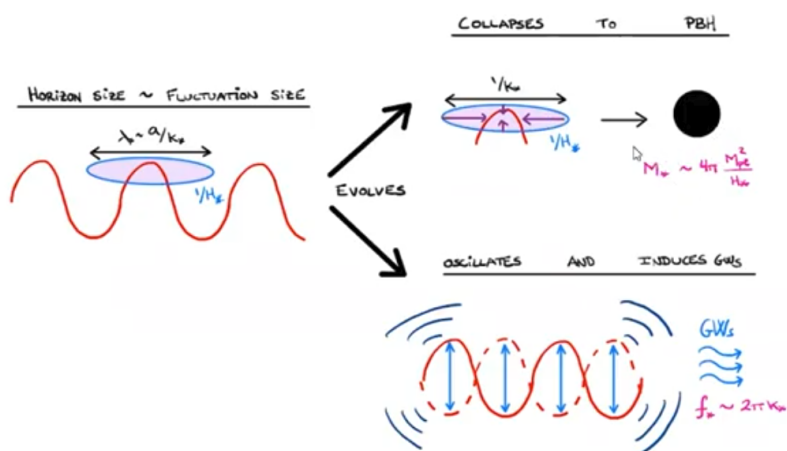


Image from: <https://domenechcosmo.netlify.app/post/induced-gws/>

Blue uncertain,
probe it with GW

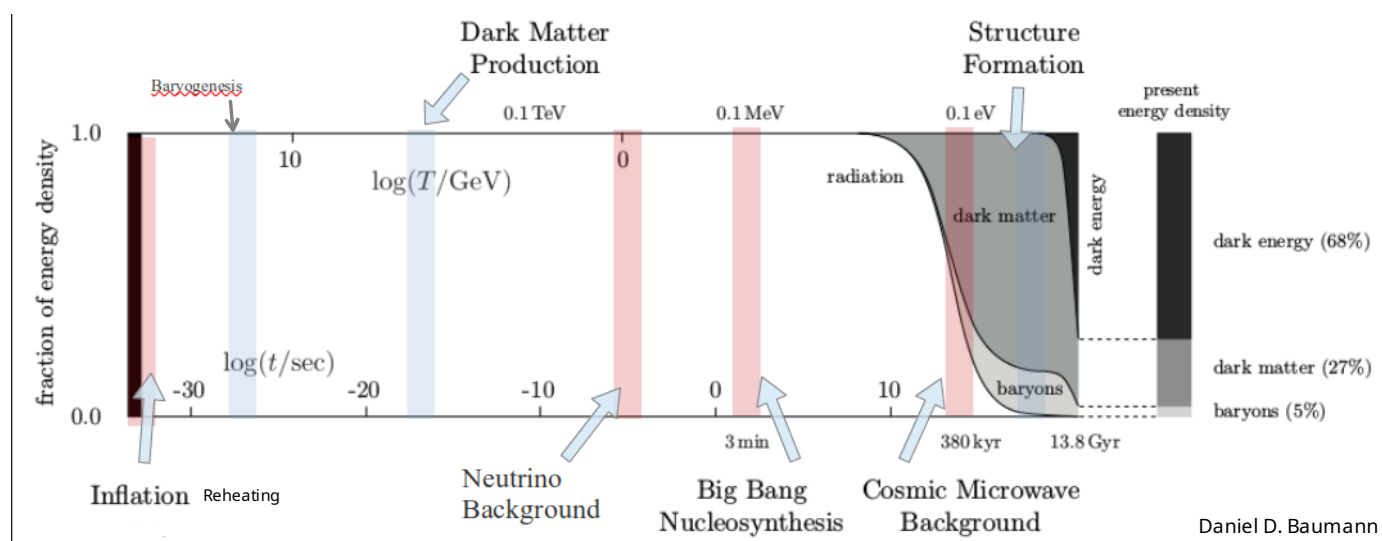
GENERATION OF GWs AND PBHs



Part III

Thermal history

Thermal history of the universe



Event	time t	redshift z	temperature T	
Inflation	10^{-34} s (?)	—	—	> Inflaton field + Reheating
Baryogenesis	?	?	?	> Particle anti-particle annihilation, small asymmetry
EW phase transition	20 ps	10^{15}	100 GeV	> Particles get mass (Higgs mechanism)
QCD phase transition	20 μ s	10^{12}	150 MeV	> Quarks and gluons combine $\rightarrow$ Baryons , Quarks and anti-quarks combine $\rightarrow$ Mesons
Dark matter freeze-out	?	?	?	> Quite early to explain its weak interaction ($t_{\text{coll}} \gg t_{\text{exp}}$)
Neutrino decoupling	1 s	6×10^9	1 MeV	> Weak interaction $e^+ + e^- \leftrightarrow \nu + \bar{\nu}$
Electron-positron annihilation	6 s	2×10^9	500 keV	> $e^+ + e^- \leftrightarrow \gamma + \gamma$
Big Bang nucleosynthesis	3 min	4×10^8	100 keV	> p, n $\rightarrow$ D, H, He, Li, (Be)
Matter-radiation equality	60 kyr	3400	0.75 eV	
Recombination	260–380 kyr	1100–1400	0.26–0.33 eV	> $e^- + p^+ \leftrightarrow \text{H} + \text{gamma}$
Photon decoupling	380 kyr	1000–1200	0.23–0.28 eV	> CMB
Reionization	100–400 Myr	11–30	2.6–7.0 meV	> $\text{H} + \text{gamma} \leftrightarrow e^- + p^+$ ★ (6)
Dark energy-matter equality	9 Gyr	0.4	0.33 meV	> Acceleration “start”
Present	13.8 Gyr	0	0.24 meV	> Life

These phases occur when certain processes are no more reversible due to the temperature drop

- e.g. neutrinos freeze-out : $e^+ + e^- \leftrightarrow \gamma + \gamma$ (reversible process) Adiabatic process
 $e^+ + e^- \rightarrow \gamma + \gamma$ become irreversible when $h\nu < m_e c^2$

Our plan

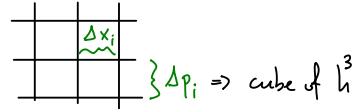
- Quantum gases : this is what we are dealing with
phase-space distribution, free energy Φ , S , P , S , N
chemical potential, μ , and reactions
- Thermal equilibrium : $T \leftrightarrow H$
- Reactions and Boltzmann equation : $\frac{T}{H}$ affects $\frac{dn_i}{dt}$ evolution in the number of particles of each type
- Neutrino decoupling : $\nu + \bar{\nu} \not\leftrightarrow e^+ + e^-$ $T_\gamma = T_\nu$
- Photons reheating : $e^+ + e^- \rightarrow \gamma + \gamma$ $T_\gamma > T_\nu$ $T_{\gamma,0} = 2,752 \text{ K}$ $T_{\nu,0} = 1,95 \text{ K}$
- Cosmological nucleosynthesis : $p, n \rightarrow D, He, Be, Li$ chain of reactions
- Recombination, photons decoupling : $p + e^- \rightarrow H + \gamma$, $e^+ + \gamma \not\leftrightarrow e^- + \gamma$ one reaction
Saha equation

Quantum statistics

Basic concepts

- Particle: $(\vec{x}, \vec{p})$ point in a 6-dimensional space = phase-space
 - it occupies a quantum energy state with energy ϵ_α α : label of the state
 - phase-space is **quantized** (also for free particles because of indeterminism principle)

$$\Delta x_i \Delta p_i \geq h \Rightarrow \text{phase-space "cells"}$$



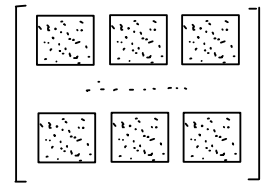
- Statistical approach:

- we do not care about each single particle, we care about ensemble properties
- Phase space distribution $f(\vec{x}, \vec{p}, t)$: probability of having a particle in a certain $(\vec{x}, \vec{p})$ cell
it can evolve as a function of time (dynamical system)

- System: collection of N particles 

- Ensemble: very large collection of identical systems
(different realizations with the same macro properties)

We consider 3 types of ensembles: Microcanonical
Canonical
Grand canonical



- Surrounding: what the system is in contact with, the thermal bath

- We consider 2-body interactions: $p_1 + p_2 \rightarrow p_3 + p_4$ e.g. $e^+ + e^- \leftrightarrow \gamma + \gamma$

Our universe: Assumptions $\text{adiathermal} \Rightarrow \text{adiabatic} \Leftarrow \text{reversible} \Leftarrow \text{no } S \text{ generation}$

- Adiabatic expansion: $\uparrow$ no heat flux because of isotropy + entropy dominated by photons (CMB)
- Thermal equilibrium: despite expansion, the system has the time to thermalize $\frac{T}{H} \gg 1$
(sometimes equilibrium is broken for some reactions)
- Ideal fluids: no long scale interactions, direct short scale interactions

System is said to be

(1) isochoric	V constant	} (1)+(2) $\rightarrow F(T, V, N)$ (Helmoltz free energy)
(2) isothermal	T constant	
(3) isobaric	P constant	

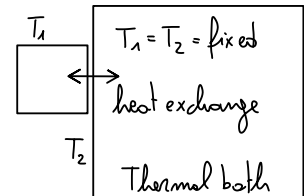
Micro-canonical ensemble

Thermodynamical isolated/closed system: no exchange of particles and energy (in form of heat/work)

i.e. fixed: $\begin{cases} \text{internal energy } U \\ \text{particle number } N \\ \text{volume } V \end{cases}$

Canonical ensemble

- Thermal equilibrium with heat reservoir at fixed T
 - Closed system with no particle exchange, exchange of energy as heat/work
- Not fixed: E energy due to the heat exchange with reservoir
Fixed: N, T, V



Probability distribution of an α -state is given by:

- Boltzmann distribution
distribution of the states

$$f_{\alpha} = \frac{e^{-\epsilon_{\alpha}/kT}}{z_c} \quad z_c = \sum_{\alpha} e^{-\epsilon_{\alpha}/kT}$$

z_c : canonical partition sum
(normalization)

- Helmoltz free energy F :

$$\frac{1}{z_c} \equiv e^{F/kT} \quad e^{-F/kT} = z_c \quad F(T, V, N) = -kT \ln(z_c)$$

statistical
mechanics
definition

$$f_{\alpha} = \frac{e^{-\epsilon_{\alpha}/kT}}{z_c} = e^{F/kT} \cdot e^{-\epsilon_{\alpha}/kT} = e^{(F - \epsilon_{\alpha})/kT}$$

$$F = U - TS$$

F is Legendre transform of internal energy U , work that a thermodynamical system can do

$$dF(T, V, N) = dU - TdS - SdT = (\cancel{TdS} - PdV + \mu dN) - \cancel{TdS} - SdT$$

Thermodynamics definition

$$= -SdT - PdV + \mu dN = \delta_T F dT + \delta_V F dV + \delta_N F dN$$

$$\left(S = -\frac{\delta F}{\delta T} \right)_V \quad \left(P = -\frac{\delta F}{\delta V} \right)_T \quad \left(\mu = \frac{\delta F}{\delta N} \right)_{T,V}$$

μ = chemical potential change in the free energy due to a change in N

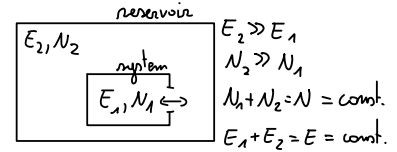
dF = change in F if you perturb T, V, N ; total differential

- At equilibrium: $F = \text{const.}$ is minimized given a T, V, N (system tend to minimize U and maximize S)

$$T = \text{const. } V = \text{const.} \Rightarrow \mu = 0, \text{ in fact: } dF = -\overset{0}{S}dT - \overset{0}{P}dV + \mu dN = \mu dN \stackrel{!}{=} 0$$

Grand canonical ensemble

- Generalization of canonical ensemble: N not fixed
- Open system, exchange particles and energy with reservoir
- Fixed: T, V, μ μ = chemical potential of a specie, describes reactions



• $N = \sum_{\alpha} N_{\alpha}$ $E(N_{\alpha}) = \sum_{\alpha} E_{\alpha} N_{\alpha}$ Tot. number of states and energy of system

• $f_{\alpha} = \frac{e^{-(E(N_{\alpha}) - \mu N_{\alpha})/kT}}{Z_{gc}}$ $Z_{gc} = \sum_{N_{\alpha}} e^{-(E(N_{\alpha}) - \mu N_{\alpha})/kT}$ grand canonical partition sum

$$= \sum_{N_{\alpha}} e^{-(\sum_{\alpha} E_{\alpha} N_{\alpha} - \mu N_{\alpha})/kT}$$

$$= \prod_{\alpha} e^{-E_{\alpha} N_{\alpha}/kT} \cdot \sum_{N_{\alpha}} e^{+\mu N_{\alpha}/kT}$$

combination of all possible states $\rightarrow$ $= \prod_{\alpha} \underbrace{\sum_{N_{\alpha}} e^{-(E_{\alpha} - \mu) N_{\alpha}/kT}}_{Z_{\alpha}}$ Partition sum of the α -state

grand canonical phase-space distribution minimizes Z_{gc}

As before: grand canonical potential ϕ

$$N = \frac{1}{kT} \frac{\delta}{\delta \mu} Z_{gc} \quad E = - \frac{\delta \ln Z_{gc}}{\delta kT} + \mu N$$

• $\phi(T, V, \mu) = -kT \ln(Z_{gc})$

• $\phi(T, V, \mu) = F(T, V, N) - \mu N = U - TS - \mu N$ ϕ is the Legendre transform of F

• $d\phi = dF - N d\mu - \mu dN = -S dT - P dV + \cancel{\mu dN} - N d\mu - \cancel{\mu dN} = -S dT - P dV - N d\mu \equiv \delta_{\mu} \phi dT + \delta_V \phi dV + \delta_{\mu} \phi d\mu$

$$S = - \left. \frac{\delta \phi}{\delta T} \right|_{V, \mu} \quad P = - \left. \frac{\delta \phi}{\delta V} \right|_{T, \mu} \quad N = - \left. \frac{\delta \phi}{\delta \mu} \right|_{T, V} \quad U = \phi + S T + \mu N = \phi + T \delta_T \phi + \mu \delta_{\mu} \phi \quad \mu = \left. \frac{\delta F}{\delta N} \right|_{T, V} = - \left. \frac{\delta \phi}{\delta N} \right|_{T, V}$$

• $\phi(T, V, \mu) = U - TS - \mu N = TS - PV + \mu N - TS - \mu N = -PV$ eq. of state

μ : energy cost/gain (change in free energy) of adding one particle
high $\mu \Rightarrow$ "costly"; resistance of the system, low $\mu \Rightarrow$ particles are "cheap"

• total energy change: $\Delta \phi = \sum_i \mu_i \Delta N_i$, if $\Delta \phi < 0 \Rightarrow$ reaction proceeds

• total chemical potential: $\mu = \sum_i \mu_i$ is conserved in reactions with constant energy, volume, number of species

• it controls reactions: transformation of specie $j = \gamma, p, \nu, \dots$ $N = \sum_j N_j$ μ_j = energy cost/gain per type
 $\mu_1 + \mu_2 = \mu_3 + \mu_4$ Chemical equilibrium $\mu = 0$: $\leftrightarrow$ (chemical potentials are related)
" > " right side has a smaller $\mu \Rightarrow$ favourite : $\rightarrow$
" < " left " " " " " $\Rightarrow$ " : $\leftarrow$

e.g. Compton scattering $e^- + \gamma \leftrightarrow e^- + \gamma + \gamma$ $\mu_e + \mu_{\gamma} = \mu_e + 2\mu_{\gamma} \Rightarrow \mu_{\gamma} = 0$

because the number of photons is not constrained

e.g. Annihilation $X + \bar{X} \leftrightarrow \gamma + \gamma$ $\mu_X + \mu_{\bar{X}} = 0 \Rightarrow \mu_X = -\mu_{\bar{X}}$

equilibrium

Case: Fermions and Bosons

$$\langle E \rangle = \phi + ST + \sum_i \mu_i N_i \text{ with more species}$$

◦ Partition sum of the α -state

Fermions : $N_\alpha = 0, 1$ (Pauli's exclusion principle) $z_\alpha = 1 + e^{-(\epsilon_\alpha - \mu)/kT}$ Fermi-Dirac (1)
 half integer Spin: $1/2, 3/2, \dots$ eg. $e^\pm, \bar{e}, n, p, \nu$

Bosons : $N_\alpha = 0, 1, 2, \dots, \infty$ $z_\alpha = (1 - e^{-(\epsilon_\alpha - \mu)/kT})^{-1}$ Bose-Einstein (2)
 integer Spin: $0, 1, 2, \dots$ eg. $\gamma, W^\pm, Z$
 (From geometric series: $\sum_{n=0}^{\infty} e^{-nX} = \sum (e^{-X})^n = (1 - e^{-X})^{-1}$)

◦ Number distribution of particles in each state

$$\begin{aligned} \langle N_\alpha \rangle &= -\frac{\delta \phi}{\delta \mu} = +kT \frac{\delta \ln(z_\alpha)}{\delta \mu} = kT \frac{1}{z_\alpha} \frac{\delta z_\alpha}{\delta \mu} \\ &= \frac{kT}{1 + e^{-(\epsilon_\alpha - \mu)/kT}} \cdot \frac{-e^{-(\epsilon_\alpha - \mu)/kT}}{+kT} = \left(\frac{e^{(\epsilon_\alpha - \mu)/kT}}{1 + e^{(\epsilon_\alpha - \mu)/kT}} \right)^{-1} \\ &= \frac{1}{e^{(\epsilon_\alpha - \mu)/kT} \pm 1} \quad \begin{array}{l} (+) \text{ fermions} \\ (-) \text{ bosons} \end{array} \end{aligned}$$

◦ Example: photons (bosons) in equilibrium

$\epsilon = h\nu$, $\mu = 0$, $g = 2$ helicity $(+1, -1)$ = Polarization internal degree of freedom
 to be multiplied to the number of states

$$N_\nu = \frac{1}{e^{h\nu/kT} - 1} \quad \text{Number distribution of states}$$

$$U(\nu) = \frac{4\pi g \nu^2 / c^3 h\nu}{e^{h\nu/kT} - 1} \quad \text{Equilibrium energy density}$$

$g_s = 4\pi g \nu^2 / c^3$ density of states, $h\nu$ energy

$$U = E_\nu g_\nu N_\nu \quad E_\nu = h\nu$$

per steradian $(1/4\pi)$, per cm^2 , per Hz

$$B_\nu(\nu) = \frac{2 h \nu^3 / c^2}{e^{h\nu/kT} - 1} \quad \text{Planck spectrum } [W \text{ str}^{-1} \text{ m}^{-2} \text{ Hz}^{-1}]$$

Ideal quantum gas

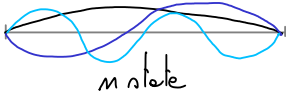
• many states $\rightarrow$ continuous: $\sum \rightarrow \int$

• Particle $\vec{p} = \hbar \vec{k} \quad \mathcal{E}(p) = \sqrt{c^2 p^2 + m^2 c^4}$

• Confined:

mode = state

3D box

$$k = \frac{2\pi}{L} m = \frac{2\pi}{V^{1/3}} m \Rightarrow m = \frac{V^{1/3}}{2\pi} k \Rightarrow d^3 m = \frac{V}{(2\pi)^3} d^3 k \quad \text{include } g: \text{ internal degrees of freedom e.g. spin}$$


$d^3 p = \hbar^3 d^3 k \rightarrow$

$$dN = \frac{gV}{(2\pi)^3} d^3 k \quad \# \text{ of states in interval of } k$$

$$= \frac{gV}{(2\pi\hbar)^3} d^3 p \quad \# \text{ " " " " " } p$$

• Continuous approximation ($\sum \rightarrow \int$)

$N = \sum_{\alpha} N_{\alpha} \quad N_{\alpha} = \frac{1}{e^{(\mathcal{E}_{\alpha} - \mu)/kT} \pm 1} \quad \text{Fermi-Dirac / Bose-Einstein}$

! $N = \int N_{\alpha} dN = \int N(p) d^3 p = \int \frac{1}{e^{(\mathcal{E}(p) - \mu)/kT} \pm 1} \frac{gV}{(2\pi\hbar)^3} d^3 p \rightarrow$

polar coord. to get density
(homogeneous) $(d^3 p = 4\pi p^2 dp) \quad \left(\frac{1}{V}\right)$

$$n = \frac{g}{(2\pi\hbar)^3} \int_0^{\infty} \frac{4\pi p^2 dp}{e^{(\mathcal{E}(p) - \mu)/kT} \pm 1} \quad \text{number density}$$

$$u = \frac{g}{(2\pi\hbar)^3} \int_0^{\infty} \frac{4\pi \mathcal{E}(p) p^2 dp}{e^{(\mathcal{E}(p) - \mu)/kT} \pm 1} \quad \text{energy density}$$

$\int \cdot \mathcal{E}(p)$

Free energy

$\Phi = -kT \ln(Z_{gc}) = -kT \ln(\prod_{\alpha} z_{\alpha}) = -kT \sum_{\alpha} \ln(z_{\alpha})$

continuous limit:

$z_{\alpha} = 1 + e^{-(\mathcal{E}_{\alpha} - \mu)/kT} \quad \text{Fermi-Dirac (1)}$
 $z_{\alpha} = (1 - e^{-(\mathcal{E}_{\alpha} - \mu)/kT})^{-1} \quad \text{Bose-Einstein (2)}$

$$\Phi(T, V, \mu) = -kT \frac{gV}{(2\pi\hbar)^3} \int_0^{\infty} 4\pi p^2 \ln(z_{\alpha}) dp$$

$$= \begin{matrix} \text{FD} \\ \text{BE} \end{matrix} kT \frac{gV}{(2\pi\hbar)^3} \int_0^{\infty} 4\pi p^2 \ln(1 \pm e^{-(\mathcal{E}(p) - \mu)/kT}) dp$$

Additionally

Pressure $P = -\Phi/V \quad \text{from } \Phi = -PV$

Entropy $S = -\frac{\partial \Phi}{\partial T}$

Number of states $N = \frac{\partial \Phi}{\partial \mu}$

Internal energy $U = \Phi + T \frac{\partial \Phi}{\partial T} + \mu \frac{\partial \Phi}{\partial \mu}$

Explicit relations of the ensemble properties

E.g. relativistic bosons in equilibrium

- relativistic $\varepsilon = cp$
- equilibrium $\mu = 0$
- bosons $z_\alpha = (1 - e^{-(\varepsilon_F - \mu)/kT})^{-1} = (1 - e^{-cp/kT})^{-1}$

$$\begin{aligned} \Phi_B(T, V, \mu) &= kT \frac{gV}{(2\pi\hbar)^3} \int_0^\infty 4\pi p^2 \ln(1 - e^{-cp/kT}) dp \\ &= \frac{(kT)^4}{c^3} \frac{gV 4\pi}{(2\pi\hbar)^3} \int_0^\infty (1 - e^{-x}) x^2 dx \\ &= -gV \frac{\pi^2}{90} \frac{(kT)^4}{(\hbar c)^3} \quad \text{free energy} \end{aligned}$$

$x \equiv cp/kT$
 $p = \frac{xkT}{c} \quad dp = \frac{kT}{c} dx$
 $p^2 dp = \left(\frac{kT}{c}\right)^3 x^2 dx$

$= -m! \zeta(m+2)$ with $m=2$ Riemann function

$$P_B = -\frac{\delta\Phi}{\delta V} = g \frac{\pi^2}{90} \frac{(kT)^4}{(\hbar c)^3} \quad \text{pressure}$$

$$s_B = -\frac{1}{V} \frac{\delta\Phi}{\delta T} = g k \frac{2\pi^2}{45} \left(\frac{kT}{\hbar c}\right)^3 \quad \text{entropy density}$$

Same for the other cases

- continuous case

- $\varepsilon_\alpha = mc^2 + p^2/2m$ 1st order relat. limit

- $z_\alpha = \frac{gV (2\pi m kT)^{3/2}}{h^3} e^{-(mc^2 - \mu)/kT}$

↓

	relativistic distributions:		non-relativistic Maxwell-Boltzmann
	Bose-Einstein	Fermi-Dirac	
number density	$n_B = g_B \frac{\zeta(3)}{\pi^2} \left(\frac{kT}{\hbar c}\right)^3$	$n_F = \frac{3}{4} \frac{g_F}{g_B} n_B$	$g \left(\frac{m kT}{2\pi\hbar^2}\right)^{3/2} e^{-kT/mc^2}$
energy density	$u_B = g_B \frac{\pi^2}{30} \frac{(kT)^4}{(\hbar c)^3}$	$u_F = \frac{7}{8} \frac{g_F}{g_B} u_B$	$\frac{3}{2} m kT$
pressure	$P_B = \frac{u_B}{3} = g_B \frac{\pi^2}{90} \frac{(kT)^4}{(\hbar c)^3}$	$P_F = \frac{u_F}{3} = \frac{7}{8} \frac{g_F}{g_B} P_B$	$m kT$
entropy	$s_B = g_B k \frac{2\pi^2}{45} \left(\frac{kT}{\hbar c}\right)^3$	$s_F = \frac{7}{8} \frac{g_F}{g_B} s_B$	

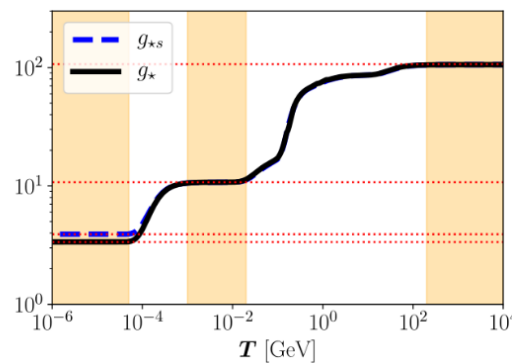
$S = s \cdot V \propto T^3 \cdot T^{-3} = \text{const.}$ Entropy is conserved

• g^* sharp transitions as relativistic species 'disappear' as $T \downarrow$ with time

$$\rho_R(T) \equiv \frac{\pi^2}{30} g_*(T) T^4 \quad s = \frac{p + \rho}{T} = \frac{2\pi^2}{45} g_{*s}(T) T^3$$

$$g_*(T) \simeq \begin{cases} 2 & T \ll m_\nu \\ 3.36 & m_\nu \ll T \ll m_e \\ 10.75 & 1 \text{ MeV} \ll T \ll 20 \text{ MeV} \\ \frac{427}{4} = 106.75 & 200 \text{ GeV} \ll T \end{cases}$$

γ
 $\gamma + \nu$
 $\gamma + \nu + e^\pm$
 all the SM



Nicolás BERNAL @ NYUAD

←
time

Fundamental particles

Table 3.2: Particle content of the Standard Model.

type		mass	spin	g
quarks	$t, \bar{t}$	173 GeV	$\frac{1}{2}$	$2 \cdot 2 \cdot 3 = 12$
	$b, \bar{b}$	4 GeV		
	$c, \bar{c}$	1 GeV		
	$s, \bar{s}$	100 MeV		
	$d, \bar{d}$	5 MeV		
	$u, \bar{u}$	2 MeV		
gluons	g_i	0	1	$8 \cdot 2 = 16$
leptons	$\tau^\pm$	1777 MeV	$\frac{1}{2}$	$2 \cdot 2 = 4$
	$\mu^\pm$	106 MeV		
	$e^\pm$	511 keV		
	$\nu_\tau, \bar{\nu}_\tau$	< 0.6 eV	$\frac{1}{2}$	$2 \cdot 1 = 2$
	$\nu_\mu, \bar{\nu}_\mu$	< 0.6 eV		
	$\nu_e, \bar{\nu}_e$	< 0.6 eV		
gauge bosons	W^+	80 GeV	1	3
	W^-	80 GeV		
	Z^0	91 GeV		
	γ	0		2
Higgs boson	H^0	125 GeV	0	1

strong charges (colors)

Strong interaction carriers (colors)

(photon)

weak interaction carriers

electromagnetic interaction carrier (charge)

QCD confinement
bind in
color neutral hadrons
(Mesons & Baryons)

unified electroweak
(no colors)

From D. D. Baumann

Composite particles

leptons do not carry color $\Rightarrow$ no strong interaction $\Rightarrow$ no QCD confinement, only electromagnetic

• Hadrons (QCD bound states: quarks, gluons)

strong force get stronger at "long distances" $\Rightarrow$ QCD confinement $\Rightarrow$ solid bound states ~ 100 MeV Binding energy
they are always color neutral

Quarks: Baryons are composed by 3 quarks (qqq) e.g. protons: $p = uud$ $npin = 1/2$
neutrons: $n = udd$ $npin = 1/2$
mesons: quark + anti quark ($q\bar{q}$) they are bosons $npin = 0, 1$

Glueballs: $g+g$ state, not observed

Tetraquarks, pentaquarks: exotic hadrons observed in LHC

Hybrid mesons: quark-antiquark + gluon excitations

• Nuclear bound states (strong force)

Systems of baryons, e.g. $D = p + n$ (deuteron), He, C, O, Fe, ...

• Electromagnetic bound states (QED)

muonic hydrogen $\mu^- + p$
positronium $e^+ + e^-$ ~ 6.8 eV
atoms, e.g. $e^- + p$ ~ 10 eV
Molecules CO_2 ~ 1 eV

Gluon-gluon interaction is possible why not photon-photon?

QED: gauge group: $U(1)$

Abelian (commutative) $F_{\mu\nu}$

field strength is linear in photon field $\Rightarrow$ No photon-photon interaction

A_μ : vector field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ $\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$, no $A_\mu A_\nu$ term!

you can have photon-charge interaction $j_\mu \dots$? no

photons do not carry a charge $\Rightarrow$ they can not interact one with another

QCD: gauge group: $SU(3)$

Non-Abelian (non-commuting generators) G_μ^a ^{color}

field strength is nonlinear $\Rightarrow$ gluon-gluon terms

$G_{\mu\nu}^a = \partial_\mu G_\nu^a - \partial_\nu G_\mu^a + g_s f^{abc} G_\mu^b G_\nu^c$ structure constants f^{abc} $[T^a, T^b] = i f^{abc} T^c$

$\mathcal{L}_{QCD} = -\frac{1}{4} G_{\mu\nu}^a G^{\mu\nu a} = -\frac{1}{4} (\partial_\mu G_\nu^a - \partial_\nu G_\mu^a + g_s f^{abc} G_\mu^b G_\nu^c) (\partial^\mu G^{\nu a} - \partial^\nu G^{\mu a} + g_s f^{abc} G^{\mu b} G^{\nu c})$

3-gluon vertexes $g_s f^{abc} G_\mu^b G_\nu^c \partial^\mu G^{\nu a}$

4-gluon vertexes $g_s^2 f^{abc} f^{def} G_\mu^b G_\nu^c G^{\mu d} G^{\nu e}$

each gluon carries a color charge indices $\Rightarrow$ they interact one with another

$\Rightarrow$ confinement : gluons interact with gluons $\Rightarrow$ force grows with length

$\Rightarrow$ asymptotic freedom: self interaction screens color at short distances $\Rightarrow$ QCD weaker at higher energy

Reactions and the Boltzmann eq.

- Reaction $P_1 + P_2 \xleftrightarrow{\text{interaction}} P_3 + P_4$

The reaction details depend on the interaction " $\leftrightarrow$ "

 - e.g. $e^+ + e^- \rightarrow \gamma + \gamma$ annihilation
 - $e^- + z^+ \rightarrow e^- + z^+$ scattering process
 - 3 or 4 body interactions are rare
- Reaction is governed by

$$T = \text{rate of collisions} \quad t_{\text{coll}} \equiv 1/T$$

$$H = \text{" " expansion} \quad t_{\text{exp}} \equiv 1/H$$
- if

$$\begin{cases} T \gg H \Rightarrow \text{particles interact} \Rightarrow \text{Thermal equilibrium} \\ T \ll H \Rightarrow \text{no interactions} \Rightarrow \text{Particles decouple / Freeze-out} \end{cases}$$
- $T = n \langle \sigma v \rangle$

n = number density $dN = n \langle \sigma v \rangle dt$ $T \equiv \frac{dN}{dt}$

σ = interaction cross section

v = relative velocity $\Rightarrow v(T)$

We need to study n, T in a cosmological context
- $\lambda = (n\sigma)^{-1}$ [m] Mean free path, $\lambda = v T^{-1}$ $R_H = c H^{-1} \Rightarrow T \ll H$ equivalent to $\lambda \ll R_H$
- Temperature evolution

$P = h/\lambda$ de-Broglie

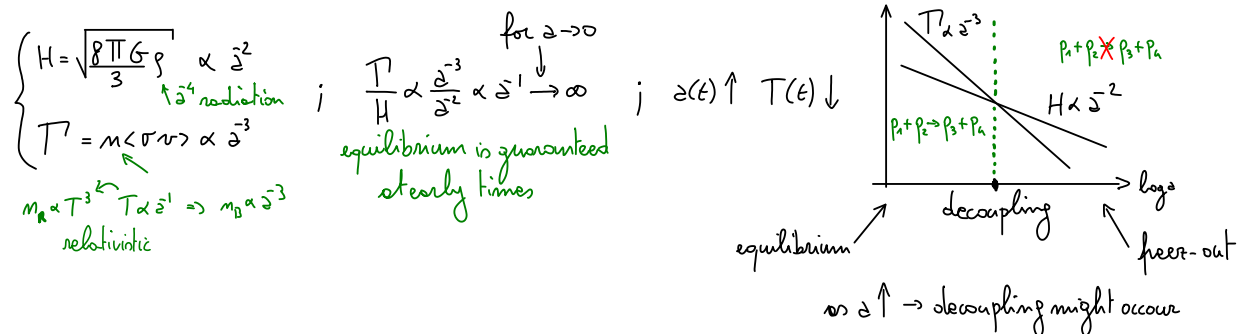
$E = k_B T$ Thermal energy

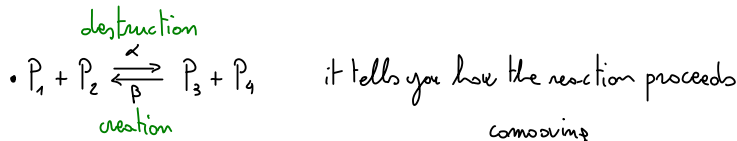
v depends on T (thermal velocities)

Non relativistic: $E = k_B T = \frac{p^2}{2m} = \frac{h^2}{2m\lambda^2} \Rightarrow T \propto \lambda^{-2}$

Relativistic: $E = k_B T = cp = \frac{hc}{\lambda} \Rightarrow T \propto \lambda^{-1}$
- Condition for equilibrium and decoupling

Early times: radiation domination, $k=0$, Λ & DE negligible $\Rightarrow H = \sqrt{\frac{8\pi G}{3}\rho}$ Friedmann eq.



Freeze-out equation / Boltzmann eq.

$\frac{1}{\partial^3} \frac{d(\overset{\uparrow}{N_1} \partial^3)}{dt} = \underbrace{-\alpha N_1 N_2}_{(1)} + \underbrace{\beta N_3 N_4}_{(2)}$
 $N_1 = m_1 \partial^3$
 $\alpha = \langle \sigma v \rangle$ thermal averaged cross-section

1) creation of P_1 particles out of reaction between P_3 and P_4

2) destruction of P_1 particles due to their reaction with P_2

3) Total change in the comoving number density of P_1 particles $\dot{N}_1 = m_1 \dot{\partial}^3$

4) Factor to convert to actual number densities, not comoving

At equilibrium $\dot{N}_1 = 0 \Rightarrow \alpha N_1 N_2 = \beta N_3 N_4$
 $\beta = \left(\frac{m_1 m_2}{m_3 m_4} \right)_{eq} \alpha$

$$= -\langle \sigma v \rangle m_1 m_2 \left[1 - \left(\frac{m_1 m_2}{m_3 m_4} \right)_{eq} \frac{m_3 m_4}{m_1 m_2} \right]$$

$$\frac{1}{m_1 \partial^3} \frac{dN_1}{d \ln \partial} = -\frac{\langle \sigma v \rangle m_2}{H} \left[1 - \left(\frac{N_1 N_2}{N_3 N_4} \right)_{eq} \frac{N_3 N_4}{N_1 N_2} \right]$$

$$\frac{d \ln N_1}{d \ln \partial} = -\frac{T}{H} \left[1 - \left(\frac{N_1 N_2}{N_3 N_4} \right)_{eq} \frac{N_3 N_4}{N_1 N_2} \right]$$

$$\frac{d}{dt} = \frac{d \partial}{dt} \frac{d}{d \partial} = a H \frac{d}{d \partial} = H \frac{d}{d \ln \partial} \quad \dot{N}_1 = H \frac{dN_1}{d \ln \partial} = (3)$$

$$T \equiv m_2 \langle \sigma v \rangle \quad \text{Reaction rate}$$

Note: $N_5 \equiv m_5 \partial^3$
 $\dot{N}_5 = \dot{m}_5 \partial^3 + m_5 3 \partial^2 \dot{\partial} = \partial^3 (\dot{m}_5 + 3 H m_5) \Rightarrow \dot{m}_5 + 3 H m_5 = -\alpha N_1 N_2 + \beta N_3 N_4$

Consequences

1) if equilibrium: $\frac{d \ln N_1}{d \ln \partial} = 0 \Leftrightarrow \frac{N_1 N_2}{N_3 N_4} = \left(\frac{N_1 N_2}{N_3 N_4} \right)_{eq}$
 N_1 does not change

2) if off equilibrium $\Rightarrow N_1$ changes by moving toward equilibrium

if $\frac{N_1 N_2}{N_3 N_4} \leq \left(\frac{N_1 N_2}{N_3 N_4} \right)_{eq} (* > 0) \Rightarrow \dot{N}_5 \leq 0$
 met destruction $\rightarrow$ $\frac{\text{source}}{\text{products}} \geq \left(\frac{\text{source}}{\text{products}} \right)_{\text{at equilibrium}}$
 met creation $\leftarrow$

3) if $T \ll H$

$\Rightarrow \dot{N}_5 \rightarrow 0$ Freeze-out (their number is conserved) reaction stops $\leftarrow$
 particles do not participate any more to the reaction,
 they leave a background (relics)
 e.g. CMB photons, neutrino background

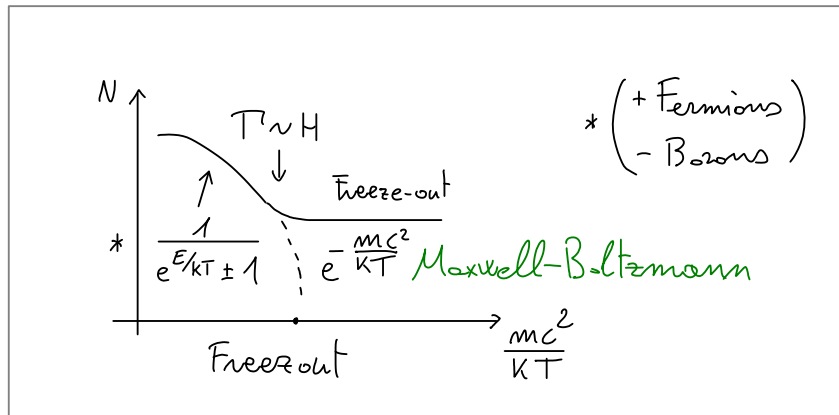
- Relativistic particles (Fermions and Bosons) $\begin{cases} n \propto T^3 \\ T \propto a^{-1} \end{cases} \Rightarrow n \propto a^{-3} \Rightarrow N \equiv a^3 n = \text{const.}$

$\frac{d \ln N}{d \ln a} \stackrel{\text{const}}{=} 0 \Rightarrow -\frac{T}{H} \left(1 - \frac{N_{\text{eq}}}{N}\right) = 0 \Rightarrow \text{i.e. } N = N_{\text{eq}} \text{ remains in T. equilibrium regardless } \frac{T}{H}!$

- Non relativistic limit

$\begin{cases} T \propto a^{-2} \\ n(T) = g \left(\frac{KT}{2\pi\hbar}\right)^{3/2} e^{-KT/mc^2} \end{cases}$ Maxwell-Boltzmann distribution $N \propto a^3 n \propto T^{3/2} T^{3/2} e^{-\frac{mc^2}{KT}} \propto e^{-\frac{mc^2}{KT}}$

$\Rightarrow \text{for } KT \ll mc^2 \text{ "cold", exponential drop} \Rightarrow N \ll N_{\text{eq}} : \frac{d \ln N}{d \ln a} = -\frac{T}{H} \rightarrow 0$



If the universe would have always remained in thermal equilibrium $\Rightarrow$ most of today's particles would be photons γ (except for baryons: baryon number is a symmetry of the Standard Model)

Understanding freeze-outs is crucial:

- γ " DM decoupling
- γ " " "
- BBN

Dark Matter Relics

Appendix

DM particles 'nowadays' are a relic

- ⇒ they broke off equilibrium
- ⇒ no more interacting with other particles

Speculations about some unknown particle X, Assume:

- e.g. Weakly Interacting Massive Particle (WIMPs) with mass M_X
 - direct interaction (in equilibrium) with Standard Model (SM) particles only in the part "l" some light (nearly massless) particles
 - "l" tightly coupled with cosmic plasma for equilibrium: $n_l = n_l^{\text{eq}}$
 - No initial asymmetry between X and $\bar{X} \Rightarrow n_X = n_{\bar{X}}$
- eg. annihilation $X + \bar{X} \leftrightarrow l + \bar{l}$

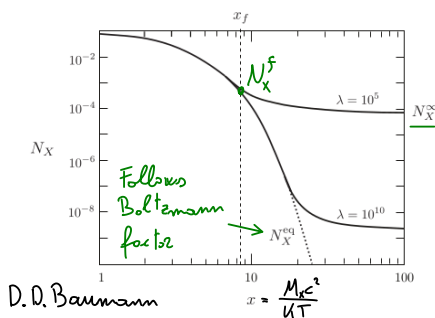
Boltzmann eq. for $X + \bar{X} \leftrightarrow l + \bar{l}$:

$$\frac{dN_X}{dt} = -\bar{\sigma} v \langle \sigma v \rangle n_X^2 \left[1 - \left(\frac{n_X}{n_X^{\text{eq}}} \right) \frac{n_l}{n_l^{\text{eq}}} \right] = -\frac{\langle \sigma v \rangle}{\bar{\sigma}^3} (N_X^2 - N_{X,\text{eq}}^2) \quad \leftarrow n_X = \bar{\sigma}^3 N_X$$

- Define $x \equiv \frac{M_X}{T}$ ($c=1=k$)
 - use it as a time variable: $\dot{x} = \frac{d}{dt} \left(\frac{M_X}{T} \right) = -\frac{M_X}{T^2} \dot{T} = -x \frac{\dot{T}}{T} \approx x H$
 - radiation domination: $H^2 = H_0^2 \Omega_r \bar{\sigma}^4 = H_0^2 \Omega_r \bar{\sigma}^4 T^4 = H_0^2 \Omega_r \bar{\sigma}^4 \frac{M_X^4}{x^4} \equiv H(M_X) x^{-4}$
- relativistic $T \propto \bar{\sigma}^{-1}$

$$\begin{aligned} \frac{dN_X}{dx} &= -\frac{\langle \sigma v \rangle}{\bar{\sigma}^3 x H} (N_X^2 - N_{X,\text{eq}}^2) \\ &= -\frac{2\pi^2}{45} g_{*s} \frac{M_X^3 \langle \sigma v \rangle}{H(M_X) x^2} (N_X^2 - N_{X,\text{eq}}^2) \\ &= -\frac{\lambda}{x^2} (N_X^2 - N_{X,\text{eq}}^2) \quad \lambda = \frac{2\pi^2}{45} g_{*s} \frac{M_X^3 \langle \sigma v \rangle}{H(M_X)} \end{aligned}$$

Riccati equation (no analytical solution)



D. D. Baumann

Figure 3.7: Abundance of dark matter particles as the temperature drops below the mass.

- in most WIMPs theories $\lambda = \text{const.}$
- Freeze-out at $x_f \sim 10$ (little dependence on M_X)
- Final abundance $N_X^\infty \equiv N_X(x=\infty)$ of dark matter
- The longer the equilibrium is kept (efficient coupling) the smaller final N_X^∞ will be

Final number density N_x^∞

Much after freeze-out : $N_x^{\text{eq}} \ll N_x^\infty \Rightarrow \frac{dN_x}{dX} \simeq \frac{\lambda}{X^2} N_x^2$

integrate $\int_{N_x^f}^{N_x^\infty} \frac{dN_x}{N_x^2} = \int_{x_f}^{\infty} \frac{\lambda}{X^2} dX \quad \frac{1}{N_x^\infty} - \frac{1}{N_x^f} = \frac{\lambda}{x_f} \quad N_x^f \gg N_x^\infty \quad \boxed{N_x^\infty \simeq \frac{x_f}{\lambda}} \quad \text{with } x_f \sim 10$

if $\lambda \uparrow$ (better interaction) $\Rightarrow N_{x\infty} \downarrow$

WIMPs Miracle

$$\Omega_x = \Omega_{\text{DM}} = \frac{\rho_{x,0}}{\rho_{c,0}} \quad \Omega_x h^2 \sim 0.1 \left(\frac{x_f}{10} \right) \left(\frac{10}{g_*(M_x)} \right)^{1/2} \overset{\text{observations}}{\downarrow} \frac{10^{-8} \text{ GeV}^{-2}}{\langle \sigma v \rangle} \simeq 0.3 \Rightarrow \sqrt{\langle \sigma v \rangle} \sim 10^{-4} \text{ GeV}^{-1} \sim 0.1 \sqrt{G_F}$$

i.e. such thermal relic has a cross section characteristic of weak interaction!

Early times

Relativistic / Non relativistic

temperature (age) we are looking into now : $T \sim 0,5 - 3 \text{ MeV}$ range

relativistic-non relativistic limit for $k_B T \gg mc^2$

$m_\gamma = 0$	$m_\mu = 105 \text{ MeV}$	Quarks u, d, s are captured within Hadrons	gluons/ quarks are locked by QCD confinement
$m_\nu \sim 0,01 - 0,1 \text{ eV}$	$m_\tau = 135 - 140 \text{ MeV}$	$W, Z, \text{Higgs} = 80 - 125 \text{ GeV}$	
$m_e = 0,511 \text{ MeV}$	$m_p = 940 \text{ MeV} = m_n$	Top, bottom, charm = $170 - 4 \text{ GeV}$	

Relativistic

Non-Relativistic

Universe domination : energy density is dominated by relativistic particles $\frac{m_b}{m_n} \sim 10^{-9}$ (Friedmann at $T \sim 1 \text{ MeV}$)

Temperature density scaling

$$g_R = \frac{\pi^2}{30} g_* \left(\frac{k_B T}{\hbar c} \right)^4 \quad \text{relativistic plasma} \quad \rightarrow \quad H^2 = \frac{8\pi G}{3} g_R \Rightarrow H(T) \quad g_m = m_i m_i \text{ non relativistic no } g_*$$

Relevant internal degrees of freedom g_*

relativistic particles at $T \sim 1 \text{ MeV} : (\gamma, e^-, e^+, \nu_e, \bar{\nu}_e, \nu_\mu, \bar{\nu}_\mu)$

$$g_\gamma = 2 \quad g_\nu = 3 \cdot 3 \cdot \frac{7}{8} = \frac{21}{8} \quad g_e = 4 \cdot \frac{7}{8} = \frac{7}{2} \quad \Rightarrow \quad g_* \approx 10,75$$

Universe is in thermal equilibrium

$T \gg H$ Reactions are efficient

Relativistic species dominator $\Rightarrow$ They are the "thermal reservoir" $\Rightarrow$ non relativistic particles adapt $\rightarrow T = T_R$

Reaction channels

Equilibrium till $T \ll H$ (efficient direct/indirect interactions) $\Rightarrow T_{\text{plasma}} = T_{e^\pm} = T_\gamma = T_\nu \quad T \geq 10 \text{ MeV}$

- $\nu_\alpha + \bar{\nu}_\beta \leftrightarrow \nu_\alpha + \bar{\nu}_\beta$
 $\nu_\alpha + \bar{\nu}_\beta \leftrightarrow \nu_\alpha + \bar{\nu}_\beta$ elastic scattering redistribute energy among flavours $\alpha = e, \mu, \tau$
- $\nu + e^- \leftrightarrow \nu + e^-$
 $\bar{\nu} + e^- \leftrightarrow \bar{\nu} + e^-$ " keeps ν in kinetic contact with the plasma through e^-, e^+
- $e^- + \gamma \rightarrow e^- + \gamma$
 $e^+ + \gamma \rightarrow e^+ + \gamma$ " redistribute energy among $e^\pm$ and γ
- $e^+ + e^- \leftrightarrow \nu + \bar{\nu}$
 $e^+ + e^- \leftrightarrow \gamma + \gamma$ annihilation/creation $T_\gamma = T_\nu$ via e^-, e^+
- $n + \nu_e \leftrightarrow p + e^-$ charged current weak process, important for BBN
 $n + e^+ \leftrightarrow p + \bar{\nu}_e$ inverse beta
 $n \leftrightarrow p + e^- + \bar{\nu}_e$ beta decay

Neutrino decoupling

Neutrinos Decoupling, Thermal freeze-out

$$e^+ + e^- \leftrightarrow \nu_e + \bar{\nu}_e$$

$$\bar{\nu}_e + e^- \leftrightarrow \nu_e + e^-$$

... weak interaction,

well below the W, Z masses
or their propagators must be included

cross section

$$\sigma_w \sim G_F^2 T^2$$

G_F : Fermi constant
 kT : characteristic energy
of the interaction

Interaction rate $\Gamma \sim m_e \langle \sigma_w v \rangle \sim G_F^2 T^5$ $m_e \sim T^3$ very steep decline

Thermal bath: $\rho_R = \frac{\pi^2}{30} g_* \left(\frac{k_B T}{\hbar c} \right)^4$ relativistic plasma

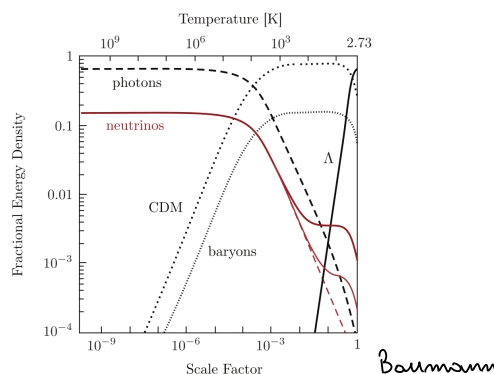
Friedmann eq.: $H = \sqrt{\frac{8\pi G}{3} \rho_R} = \sqrt{\frac{8\pi^3 G}{90} g_*} \left(\frac{k_B T}{\hbar c} \right)^{3/2}$

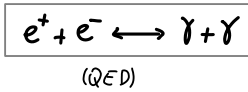
Decoupling (freeze-out) when $H \approx \Gamma$: $\sim 1 \text{ MeV}$ (precisely $0,8 \text{ MeV}$) $\frac{\Gamma}{H} \sim \left(\frac{T}{1 \text{ MeV}} \right)^3$

thermal freeze-out $\Rightarrow$ ν are free to travel along geodesics: Neutrino Background (CνB)
Not yet detected

Properties of CνB

- by decoupling from e^-, e^+ , ν loose the indirect coupling to γ ! $\Rightarrow$ temperatures, T_γ, T_ν , can evolve independently
- but $T_\nu = T_\gamma$ keep the name because ultra-relativistic: $T_\nu \propto a^{-1}$ as radiation
- phase-space distribution freezes: $f_\nu(p) = \frac{1}{e^{p/T_\nu} + 1}$ redshifting as radiation
- Fermi-Dirac distribution: $n_\nu = \frac{3\zeta(3)}{4\pi^2} T_\nu^3$ $N_\nu \equiv n_\nu \cdot a^3 \propto M_\nu T^{-3} = \text{const.}$ comoving number density
their number is conserved (no more creation/annihilation)
Boltzmann eq. $\frac{d(n a^3)}{dt} = 0$
- Today: $n_{\nu, \text{tot}, 0} = 336 \text{ cm}^{-3}$ (for the 3 flavours) $\sim n_{\gamma, 0} = 411 \text{ cm}^{-3}$
 $\rho_\nu = \sum_i m_{\nu_i} n_{\nu_i}$ $\sum_i m_{\nu_i} c^2 < 1 \text{ eV}$ upper bound clustering, CMB, SN-1a
 $0,001 < \Omega_{\nu, 0} < 0,02$
 $T_{\nu, 0} = T_{\text{CMB}, 0} \left(\frac{4}{11} \right)^{1/3} = 1,95 \text{ K} = 0,17 \text{ MeV}$ (see next page)
- not perfectly instantaneous decoupling $\Rightarrow$ slight heating occurs $N_{\text{eff}}^{\text{SM}} = 3,046$
- free streaming component (Hot, $m_\nu c^2 \ll kT$): effects cosmic structure formation, more on that later

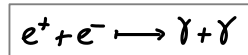


Electron-Positron annihilation

• bi-directional until $\langle h\nu \rangle < 2m_e c^2 \approx 1 \text{ MeV}$ Right after ν decoupling

$n_{e^\pm} \propto e^{\frac{-m_e c^2}{k_B T}}$ "Boltzmann suppressed" energies above threshold become more than the process becomes irreversible $\rightarrow$ freeze out

γ can no more create e^- or e^+ because their energy is too low (redshift)



• all energy (entropy) goes into photons with $\langle h\nu \rangle \sim m_e c^2 > \langle h\nu_{\text{old}} \rangle$

$\Rightarrow$ Reheating of radiation

↑ photon frequency

full annihilation

$$S_{e^+} + S_{e^-} + S_\gamma = S'_\gamma$$

before after

• Entropy is conserved: $S_F = \frac{7}{8} \frac{g_F}{g_B} S_B$ with $g_{e^+} = g_{e^-} = g_\gamma = 2 \Rightarrow$

$$\frac{S_F}{S_B} = \frac{7}{8}$$

Relativistic distri.

(e^-, e^+)

(γ)

internal degrees of freedom

$$\left(\frac{7}{8} + \frac{7}{8} + 1 \right) S_\gamma = S'_\gamma$$

11/4

$$S_B = g_B K \frac{2\pi^2}{45} \left(\frac{kT}{\hbar c} \right)^3$$

$$\Rightarrow 1,5 T_\gamma \approx T'_\gamma$$

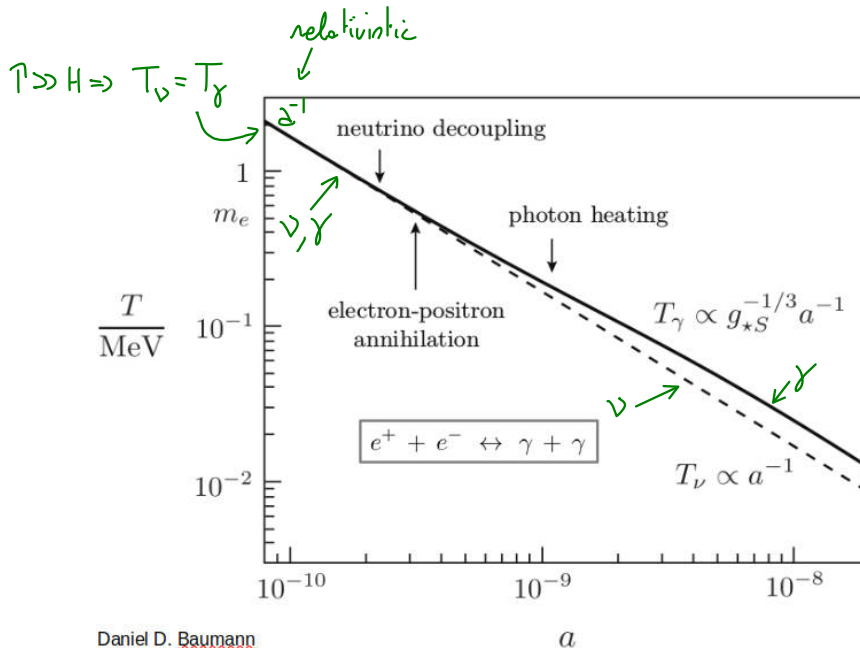
$(11/4)^{1/3}$

(measured)

Today: $T_\gamma = 2,752 \text{ K}$

$\Rightarrow T_\nu = 1,95 \text{ K}$ (expected)

photons are hotter
"Reheating"



* recall: $S = s a^3 \propto T^3 \cdot T^{-3} \propto \text{const.}$ Entropy is conserved

Effective number of relativistic species

N_{eff} : effective number of neutrino species

- Standard model of particles: $N_\nu = 3$, $\nu_e, \nu_\mu, \nu_\tau \Rightarrow N_{\text{eff}}^{\text{int}} = 3$ for instantaneous reheating but...
- after reheating $T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma$
 $\rho_{\text{rad}} = \rho_\gamma \left[1 + \frac{7}{8} \left(\frac{4}{11}\right)^{4/3} N_{\text{eff}} \right]$ total radiation energy density, effect on expansion rate
 expansion as if there are $N_{\text{eff}} = 3.046$ fully relativistic neutrino-like degrees of freedom at $T \sim 1 \text{ MeV}$
- Why? not perfectly instantaneous decoupling
 at reheating $e^+ + e^- \rightarrow \gamma + \gamma$, not all ν are decoupled $\Rightarrow$ they get some entropy/heat (and photons less)
 partial ν heating ~ 0.034
 QED corrections ~ 0.012 } 0.046
 small correction to $T_\nu/T_\gamma = \left(\frac{4}{11}\right)^{1/3} = 0.714 \rightarrow 0.716$
 slight enhancement of $N_{\text{eff}} = 3.046$
- N_{eff} useful because γ are observable (CMB), ν not
- N_{eff} can be affected by other possible relativistic species before recombination mixing N_{eff}
 $\Rightarrow$ Deviations from $N_{\text{eff}} = 3.046$ would signal possible new light physics

Could neutrinos be the famous dark matter?

- Density of neutrinos $\rho_\gamma, \rho_\nu \sim T^4 \Rightarrow \rho_{\nu,\text{tot}} = \rho_{\nu+\bar{\nu}} = \frac{7}{8} N_\nu g_\nu \left(\frac{4}{11}\right)^{4/3}$
- $N_\nu g_\nu \approx 3 \pm 0.3$ (CMB measures)
 3 families in standard model of particles and $g_\nu = 1 \Rightarrow \rho_{\nu,\text{tot}} \approx 0.68 \rho_\gamma$
- There are $2N_\nu$ species of neutrinos, ν , and anti-neutrinos, $\bar{\nu}$, and $m_\nu = \alpha T^3$ $\alpha = \text{const.}$
 $\Rightarrow m_{\nu^*} = \frac{3}{4} (2N_\nu) m_\nu(T_\nu) = \frac{3}{2} N_\nu \alpha T_\nu^3 = \frac{3}{2} N_\nu \alpha T_\gamma \left(\frac{T_\nu}{T_\gamma}\right)^3 = \frac{6}{11} N_\nu m_\nu(T_\gamma)$ $m_{\nu^*} = 340 \frac{1}{\text{cm}^3}$
 $m_{\nu^*} = m_{\nu^*} + m_{\bar{\nu}}$
- $\Rightarrow$ today energy density $\rho_{\nu,0} = \frac{6}{11} m_\nu \sum m_\nu$
 $\Omega_{\nu,0} = \frac{8\pi G}{3H_0^2} \frac{6}{11} m_\nu \sum m_\nu \stackrel{!}{=} 0.3$ with $H_0 \approx 70 \Rightarrow \underline{m_\nu = \text{SeV}}$

But... neutrino mass is $m_\nu < 1 \text{ eV}$ (measured from solar neutrino oscillations + cosmic structure formations)

No! too large $\Rightarrow$ Neutrinos alone can not compose all dark matter!

Reactions and Saha's equation

- At early times photons are energetic enough (T is large) to provide an efficient photon dissociation
- " " " we deal with a plasma: n, p, e^- tightly coupled with γ in thermal equilibrium
- Assumptions: ideal gases composed by indistinguishable particles low enough densities such to ignore degeneracies

Baryon-Photons ratio η is key

- Number density of baryons today, assume they are all in hydrogen $H \Rightarrow m_p$ protons

$$n_B = \frac{\rho_B}{m_p} = \frac{\Omega_B}{m_p} \rho_c \approx \frac{\Omega_B}{m_p} \frac{3H^2}{8\pi G} \approx 1,1 \cdot 10^{-5} \Omega_B h^2 \text{ cm}^{-3} \quad \Omega_B = 0,04 \text{ measured}$$

- Number of photons (bosons) $n_\gamma = g_r \frac{\zeta(3)}{\pi^2} \left(\frac{kT_0}{hc} \right)^3 = 407 \text{ cm}^{-3} \quad T_0 = T_{CMB} = 2,752 \text{ measured}$

- Baryons - Photons fraction $n_p = \frac{3}{4} \frac{\rho_p}{\rho_B} n_B \Rightarrow \eta \equiv \frac{n_B}{n_\gamma} = 2,7 \cdot 10^{-8} \Omega_B h^2 = \text{const.} !$

$\Rightarrow$ for each baryon you have $> 10^8$ photons

$\Rightarrow$ Photons set the temperature $T = T_\gamma$!

$\Rightarrow$ entropy of universe is dominated by photons

irreversible processes are negligible $\Rightarrow$ adiabatic expansion !

η is important because it tunes nucleosynthesis and recombination !

As $\eta \uparrow$, temperature $T \downarrow$ certain recombination processes take place

e.g. $p + n \rightarrow D + \gamma, \dots$ "cosmological nucleosynthesis" $\Rightarrow$ formation of D, He, Be, Li

e.g. $p + e^- \rightarrow H + \gamma$ "recombination" $\Rightarrow$ neutral atoms

Initially in equilibrium $\rightarrow$ Saha equation

Note: why are there baryons at all ?!

during their creation they should have annihilated with anti-baryons: $B^+ + B^- \rightarrow \gamma + \gamma$

$\Rightarrow$ must have been asymmetry in baryon / anti-baryon number,
possible under Sakharov conditions, e.g. CP violation

Saha equation

Boltzmann eq., consider equilibrium $\dot{n}_i + 3H n_i = C_i$

$\Gamma \gg H \rightarrow$ thermal (chemical) equilibrium $C_i = 0 \Rightarrow \dot{n}_i^{eq} \Rightarrow n_i^{eq} \quad i = p, e, H, \dots$

Consider : $i + j \leftrightarrow k + \gamma$

Chemical equilibrium : $\mu_i + \mu_j = \mu_k + \mu_\gamma \quad \mu_\gamma = 0$

Non relativistic limit : $n_i^{eq} = g_i \left(\frac{m_i K T}{2\pi \hbar^2} \right)^{3/2} e^{(\mu_i - m_i c^2)/KT}$ Maxwell-Boltzmann distribution $\epsilon_i \approx m_i c^2 + \frac{p^2}{2m_i}$
 $= \lambda^3 g_i (m_i T)^{3/2} e^{(\mu_i - m_i c^2)/KT}$ $\lambda^3 \equiv (KT_p)/(2\pi \hbar^2)$

We are talking about reaction rates

combination rate $R_{form} \propto n_i n_j$

ionization " $R_{ion} \propto n_k$ equilibrium $R_{form} = R_{ion}$

$X(T) = \frac{n_i^{eq} n_j^{eq}}{n_k^{eq}}$ degree of dissociation of "k" in equilibrium (some "k" form, some get ionized)

γ are the thermal bath setting $T = T_\gamma$

Saha equation : $\frac{n_i^{eq} n_j^{eq}}{n_k^{eq}} = \frac{g_i g_j}{g_k} \lambda^3 \left(\frac{m_i m_j}{m_k} \right)^{3/2} T^{3/2} e^{(\mu_i + \mu_j - \mu_k - m_i c^2 - m_j c^2 + m_k c^2)/KT}$

general $\rightarrow \frac{g_i g_j}{g_k} \left(\frac{KT}{2\pi \hbar^2} \right)^{3/2} \left(\frac{m_i m_j}{m_k} \right)^{3/2} e^{-E_i/KT}$ $E_i \equiv (m_i + m_j - m_k) c^2$ Binding energy

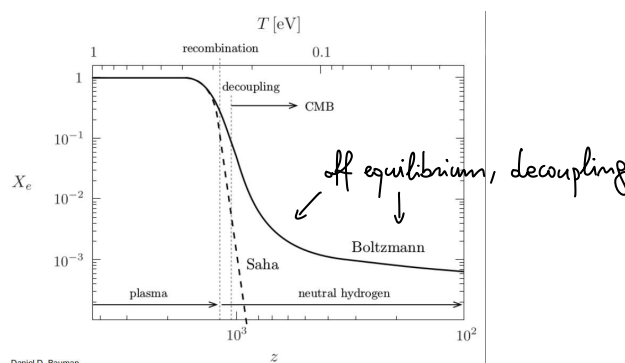
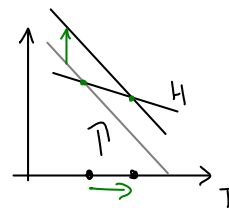
$\frac{n_i^{eq} n_j^{eq}}{n_k^{eq}} = \frac{m_B^3 X_i X_j}{m_B^3 X_k} \Rightarrow \frac{X_i X_j}{X_k} = \frac{g_i g_j}{m_B g_k} \left(\frac{KT}{2\pi \hbar^2} \right)^{3/2} \left(\frac{m_i m_j}{m_k} \right)^{3/2} e^{-E_i/KT}$ $X_i \equiv \frac{m_i}{m_B}$ dissociation fraction (free i-th particle / baryons)
 $= \frac{g_i g_j}{m_B g_k} \frac{\pi^2}{2 \zeta(3)} \left(\frac{m_e c^2}{KT} \right) e^{-E_i/KT}$ $m_B = \eta m_p = \eta 2 \frac{\zeta(3)}{\pi^2} \left(\frac{KT}{\hbar c} \right)^3$ Baryon fraction η

It is a delayed reaction

$\Gamma_i \approx H$ reaction stops: η "delays" this equality

$\Gamma_i = n_j \langle \sigma v \rangle = X_j m_B^3 \langle \sigma v \rangle \propto \eta$

Timing is not just set by $e^{-E_i/KT}$ i.e. dissociation rate $E_i \leq KT$ thermal energy



Cosmological nucleosynthesis

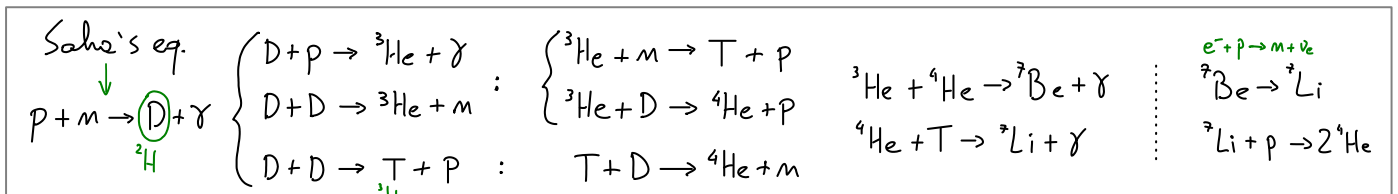
- P and n form at QCD phase transition (binding of quarks) $kT_{\text{QCD}} \sim 160 \text{ MeV}$
- Equilibrium thanks to reactions such as $n + \nu_e \rightleftharpoons p + e^-$
equilibrium Boltzmann distribution $n_p > n_n$ $\frac{n_n}{n_p} = e^{-(m_n - m_p)c^2/kT} \approx \frac{1}{6}$ $m_n > m_p$ $\Delta mc^2 = 1,4 \text{ MeV}$
- At $T_0 \sim 0,8 \text{ MeV} \Rightarrow$ thermal freeze-out: n, p number density freezes
- Spontaneous decay of n, decay time: lifetime $\sim 900 \text{ s} \Rightarrow$ little drop in n_n : $\frac{n_n}{n_p} = 1/7$ at $kT \sim 0,1 \text{ MeV}$
- Binding energy of nuclei $\sim 2 \text{ MeV}$ $\Rightarrow$ for $k_b T > 2 \text{ MeV}$ p and n can not form bound states immediate photon-dissociation
Afterwards they can combine

Helium mass fraction abundance, rough estimate:

assume all n bind in ${}^4\text{He}$ nuclei $\Rightarrow$ number of ${}^4\text{He}$ atoms $\left(\frac{p}{n} \right) = \left(\frac{M_n}{2} \right)$ $\left. \vphantom{\frac{p}{n}} \right\}$ $Y = \frac{4 \left(\frac{M_n}{2} \right)}{(M_n + M_p)} \approx 0,25$
 ${}^4\text{He}$ contains 4 nucleons $\leftarrow M_B$

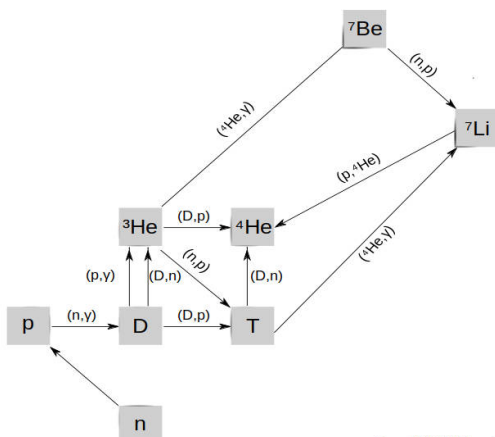
Helium abundance, how to do it:

- Need to form ${}^4\text{He}$ nuclei $\left(\frac{p}{n} \right)$ $\begin{matrix} p & p \\ \searrow & \swarrow \\ n & n \end{matrix}$ 4-body interaction not likely to occur
- The number of neutron is crucial
- Only 2 body interactions are likely to occur $\Rightarrow$ need to go through intermediate steps
Need Deuterium, $D = {}^2\text{H}$, to build up ${}^4\text{He}$



$$p + n \rightarrow D + \gamma \quad \frac{X_p X_n}{X_D} = \frac{g_p g_n}{g_D} \frac{\pi^2}{2 \zeta(3)} \left(\frac{m_e c^2}{kT} \right) e^{-E_i/kT} = \frac{2 \pi^2}{3 \zeta(3)} \left(\frac{m_e c^2}{kT} \right) e^{-E_i/kT}$$

$E_i = 2,22 \text{ MeV}$ $g_D = 3$ $g_p = g_n = 2$
binding energy > 0 for bound nucleus



Credit Wikipedia

These reactions are all interconnected

Monte Carlo numerical evaluation
consider cross section σ of each reaction

Strongest dependencies

- η is crucial: it tunes the entire process

Saha eq.: retarded reaction, D forms at $kT = 0,08 \text{ MeV}$, $t = 3 \text{ min}$, $z \sim 10^8$ (Binding energy $\sim 2 \text{ MeV}$)
 γ roles! photo dissociation,

- n_m is crucial: all n bind $\Rightarrow$ D production stops

- D is crucial $\left\{ \begin{array}{l} \text{Too much D} \Rightarrow \text{too few free } n \text{ to proceed with the reaction} \\ \text{Too few D} \Rightarrow \text{the chain can not proceed because of its lack} \end{array} \right\}$ reaction rate about right
 t^{-1} from H, $\langle \sigma v \rangle$ is known $\Rightarrow n_B$ constraint!
 $: n_B \langle \sigma v \rangle \approx t^{-1}$
 (Gamow criteria)

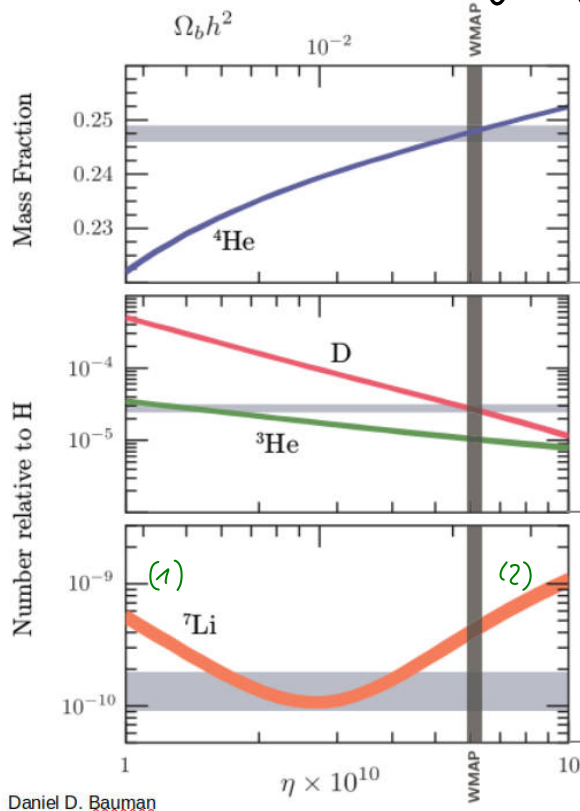
- Expansion rate H is crucial: the faster, the less time to process the material

Only light elements form, not enough time to proceed to form heavier elements and... $\Rightarrow$ freeze-out

- Elements with atomic number $A=6,8$ are unstable $\Rightarrow$ additional difficulty in making heavier elements

- The entire universe is like the interior of a star for a short time
 in contrast, stars keep the plasma combined because of gravity $\Rightarrow$ heavier elements

Amount of the products: dependency on η



Saha eq.: η tunes the entire process

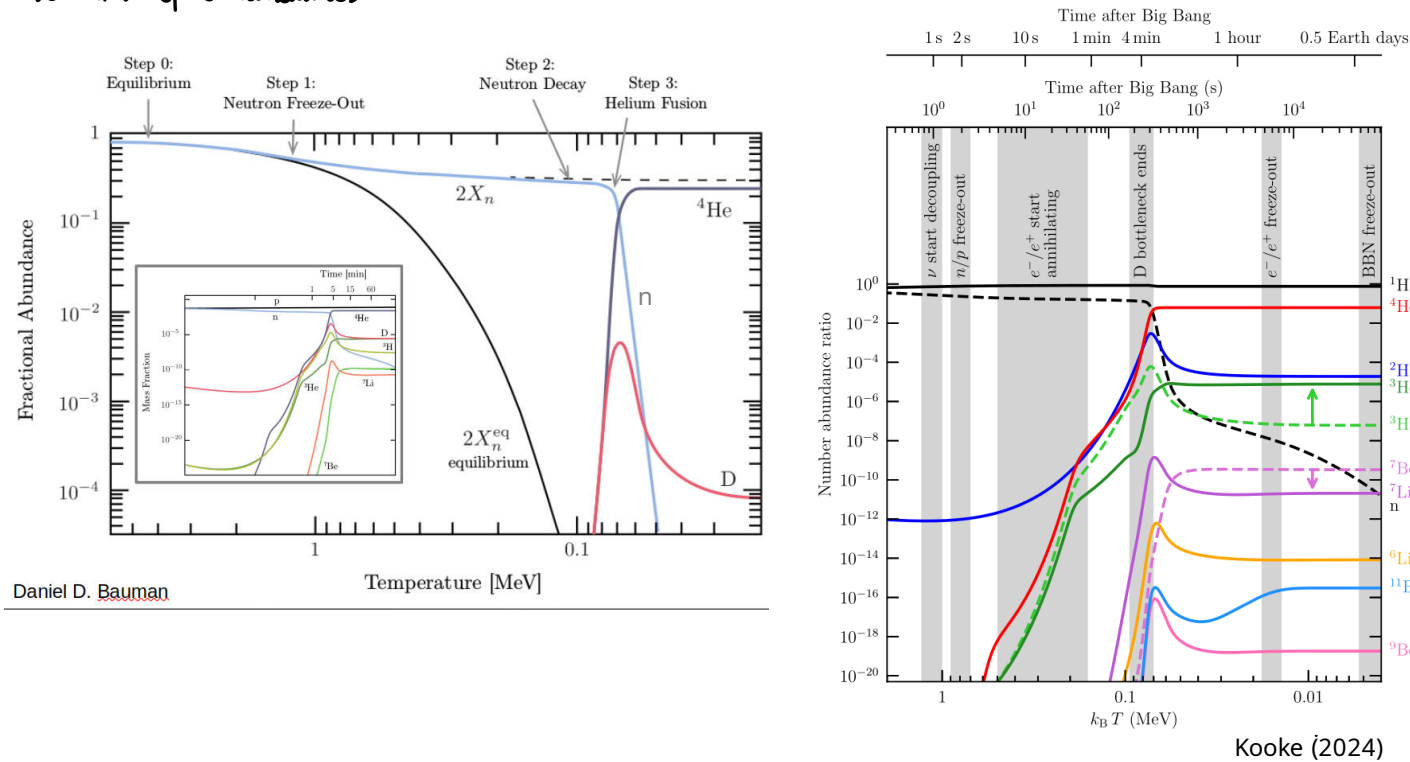
if $\eta \uparrow \Rightarrow$ BBN starts earlier $\Rightarrow \gamma \uparrow$
 more neutrons (that did not decay)

if $\eta \uparrow$ (more time for fusion) $\Rightarrow$ D, $^3\text{He} \downarrow$
 because D, ^3He burned by fusion

if $\eta \uparrow$ (more time for fusion) $\Rightarrow$ $^7\text{Be} \uparrow \Rightarrow$ $^7\text{Li} \uparrow$ (1)
 because produced by fusion ($^7\text{Be} \rightarrow ^7\text{Li}$)

if $\eta \downarrow$ less photo dissociation for $^7\text{Li} \Rightarrow$ $^7\text{Li} \uparrow$ (2)

Evolution of abundances



All other elements come from the stars and humans

Periodic Table of the Elements																		19																																																																																																																																								
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Constraints on Ω_b (m_b)

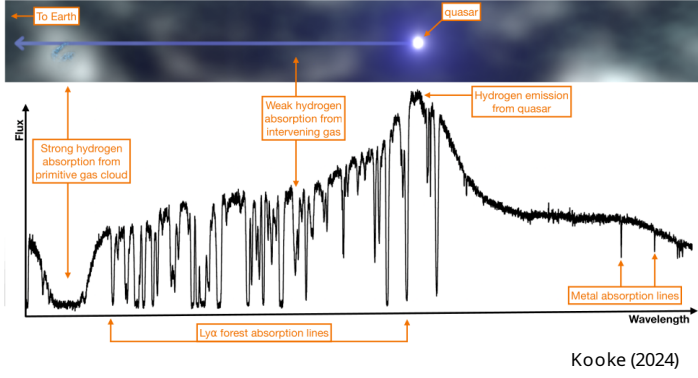


Fig. 4 The top image shows a quasar with its emission directed towards our telescopes on Earth. The space in between is filled with galaxies and intergalactic gas that absorb the quasar light at wavelengths corresponding to atomic transitions (see the corresponding spectrum in the bottom panel). The expansion of the Universe causes these transitions to redshift relative to each other, leading to a forest of absorption lines at wavelengths less than the hydrogen Ly- α emission of the quasar. Most of these absorption lines are due to weak hydrogen absorption. By pure chance, sometimes a large overdensity of primitive gas is intersected. This primitive gas cloud imprints the Lyman series absorption lines of deuterium and hydrogen on the quasar spectrum, and we can use these absorption lines to count the number of deuterium and hydrogen atoms that are in the gas cloud. In addition to deuterium and hydrogen, some heavy elements (i.e. 'metals' made by stars, such as carbon and oxygen) absorb the quasar light, allowing us to assess the pollution of this gas since BBN.

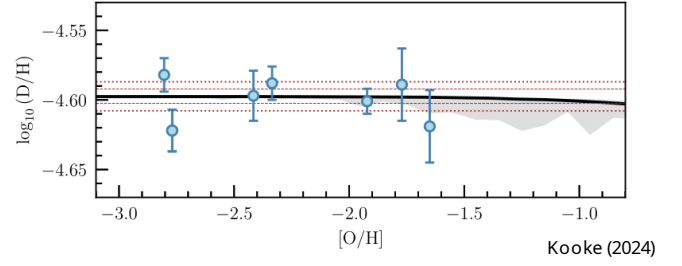
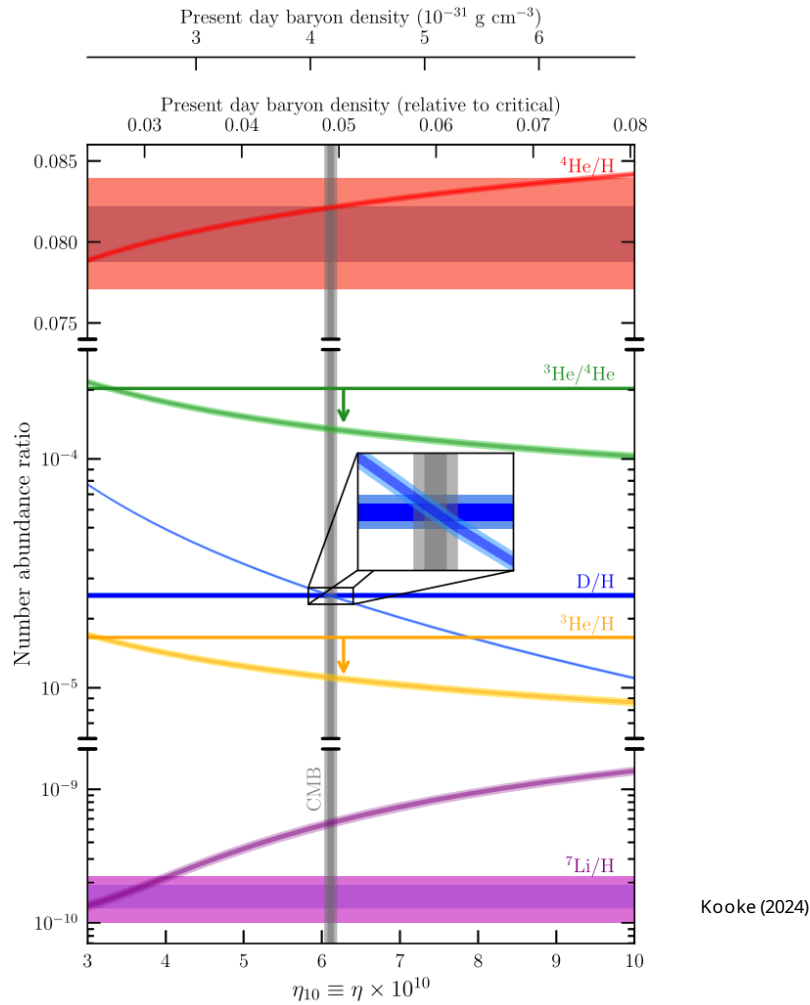


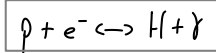
Fig. 7 Seven independent measures of the deuterium abundance (i.e. the relative number of deuterium to hydrogen atoms) of gas clouds (blue symbols with error bars; [Cooke et al. 2018](#)). The x-axis shows the oxygen abundance, displayed on a log scale relative to solar (i.e. -2 is one-hundredth of the number of oxygen atoms in the sun relative to hydrogen, while -3 represents one-thousandth of the number of oxygen atoms). All seven measures are consistent with each other. The red horizontal lines represent 68 and 95 per cent confidence interval of the weighted mean value of these seven measures. The black line shows an example model of the chemical evolution of deuterium ([van de Voort et al. 2018](#)), where the 3σ region is shown by the light grey band. Note that near-pristine systems have a deuterium abundance that is consistent with the primordial value to within < 0.1 percent, based on this calculation.



Recombination and photons decoupling

Recombination

Consider



γ are the thermal bath setting $T = T_\gamma$

$T \gg H \rightarrow$ thermal (chemical) equilibrium $\rightarrow$ Saha equation

Saha equation: $\frac{n_e n_p}{n_H} = \frac{g_e g_p}{g_H} \frac{\pi^2}{2 \zeta(3)} \left(\frac{m_e c^2}{kT} \right)^2 e^{-E_i/kT}$ $E_i = (m_p + m_e - m_H) c^2 = 13.6 \text{ eV}$ $m_p \approx m_H$ neglect binding energy

$\frac{n_e^2}{n_H} = \frac{n_B X_e^2}{1 - X_e} \Rightarrow \frac{X_e^2}{1 - X_e} \approx \frac{0.26}{\eta} \left(\frac{m_e c^2}{kT} \right)^2 e^{-E_i/kT}$

$m_e = m_p$ $g_e = 2$ $g_p = 2$ $g_H \approx 2 \cdot 2$ spins

$X_e = \frac{n_e}{n_B}$ ionization fraction (free electrons versus baryons)

$\frac{n_e^2}{n_B} = \frac{n_B^2 X_e^2}{n_B}$

$n_B = n_p + n_H = n_e + m_H$ $n_H = n_B - n_e = n_B (1 - X_e)$

* unresolvable hyperfine splitting (21cm) negligible as $T \sim 3000 \text{ K}$

End of recombination

• End when most of e, p are bound in hydrogen:

i.e. low degree of ionization $X = \frac{n_e}{n_B} \ll 1 \Rightarrow \frac{X^2}{1-X} \approx X^2 \propto e^{-X/kT} \ll 1 \Rightarrow kT \ll X$ (γ are "cold" enough)

$T_0 = 2,75 \text{ K}$ $T_{\text{rec}} \approx 3500 \text{ K} \approx 0,3 \text{ eV}$ for $x=0,1$
 $T_{\text{rec}} = T_0 \Delta^{-1} = T_0 (1+z_{\text{rec}}) \Rightarrow z_{\text{rec}} = 1320$ after radiation-matter equality $z_{\text{eq}} = 3500$

matter domination $a(t) = (t/t_0)^{2/3} \Rightarrow t_{\text{rec}} = t_0 (1+z_{\text{rec}})^{-2/3} = 290000 \text{ yr}$

• This process is not instantaneous, it lasts for some time $\delta T \approx 200 \text{ K}$ solve Boltzmann eq.

• Approximation breaks when $T \sim H \Rightarrow C_1 \neq 0$

Recombination is "delayed" "Delayed reaction"

• End happens much after ionization energy $E_i = 13,6 \text{ eV} \sim 10^5 \text{ K}!$

because of the very large number of photons $\eta = \frac{n_B}{n_\gamma} = 2,7 \cdot 10^{-8} \Omega_B h^2$ This is also why γ are the thermal bath

$E_\gamma = h\nu$ follows Planck spectrum $\Rightarrow$ small fraction have $h\nu > E_i$ but n_γ is large (compensating)

In reality the process is even slower than what predicted by Saha's eq.

- the reaction is $e^- + p^+ \leftrightarrow H + \gamma$ captured e^- transition to ground state
 Lyman- α : $2P \rightarrow 1S \Rightarrow h\nu \geq E_{Ly\alpha} = \frac{32}{4} = 10,2 \text{ eV}$

- Reaction proceeds only if lower ν photons are emitted:

forbidden transition $2S \rightarrow 1S$ (2 photons are emitted)

low probability of transition $\Rightarrow$ slow $\Rightarrow$ source of a further delay Peebles eq. accounts for that

Photons decoupling

Drops of $m_e \Rightarrow e^- + \gamma \leftrightarrow e^- + \gamma$ Thomson scattering $\Gamma_\gamma \simeq n_e \sigma_T$ as $m_e \rightarrow 0 \Rightarrow \Gamma_\gamma \rightarrow 0$

$\Gamma(T_{\text{dec}}) \sim H(T_{\text{dec}})$ Freeze-out / decoupling $\Rightarrow \gamma$ relic

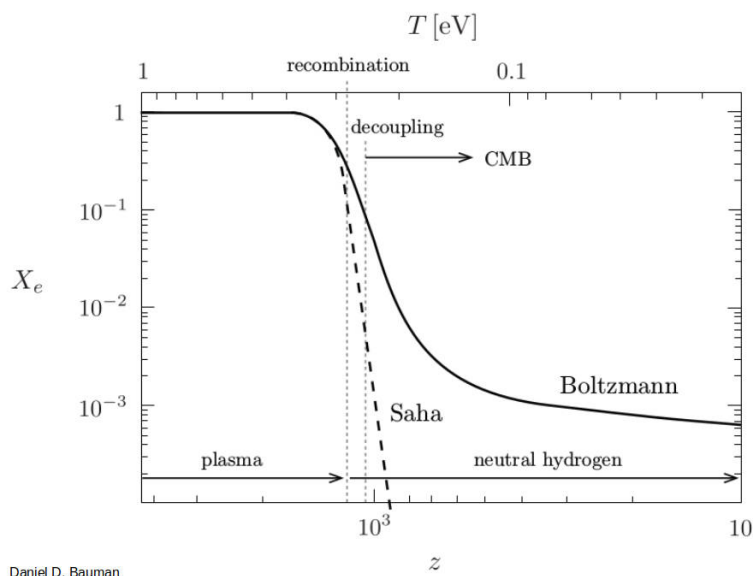
photons relic background: **Cosmic Microwave Background (CMB)** = last scattering surface

$\rightarrow \lambda$ of these γ today

$$\Gamma_\gamma(T_{\text{dec}}) \simeq n_e \sigma_T = n_b X_e(T_{\text{dec}}) \sigma_T = 2\eta \left(\frac{4}{\pi^2} \right) \left(\frac{kT_{\text{dec}}}{hc} \right)^3 X_e(T_{\text{dec}}) \sigma_T \quad \left\{ \begin{array}{l} T_{\text{dec}} = 0.27 \text{ eV} \\ z_{\text{dec}} \simeq 1100 \\ t_{\text{dec}} \simeq 380000 \text{ yr} \end{array} \right.$$

$\leftarrow$ Saha

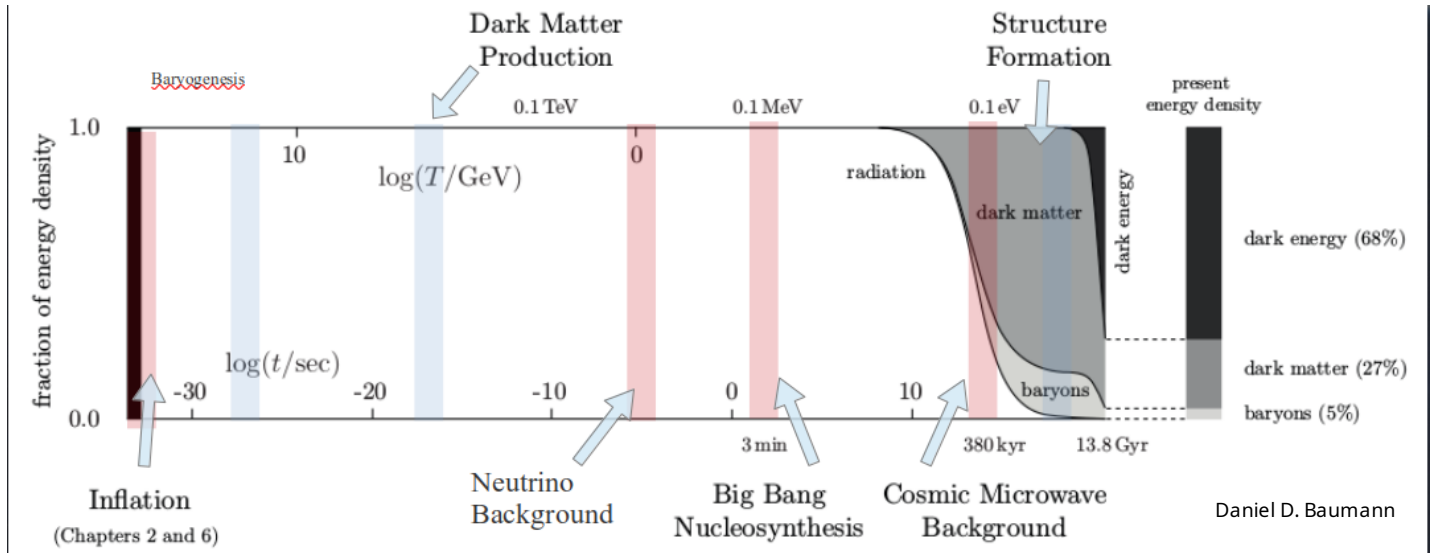
$$H(T_{\text{dec}}) = H_0 \sqrt{\Omega_m} \left(\frac{T_{\text{dec}}}{T_0} \right)^{3/2}$$



Summary

The chronological:

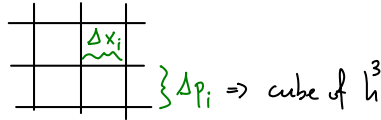
- 1) ν decoupling / photons reheating
- 2) Big-Bang nucleosynthesis
- 3) recombination and photons-matter decoupling



Event	time t	redshift z	temperature T	
Inflation	10^{-34} s (?)	–	–	> Inflaton field
Baryogenesis	?	?	?	> Particle anti-particle annihilation, small asymmetry
EW phase transition	20 ps	10^{15}	100 GeV	> Particles get mass (Higgs mechanism)
QCD phase transition	20 μ s	10^{12}	150 MeV	> Quarks and gluons combine $\rightarrow$ Baryons , Quarks and anti-quarks combine $\rightarrow$ Mesons
Dark matter freeze-out	?	?	?	> Quite early to explain its weak interaction ($t_{\text{coll}} \gg t_{\text{exp}}$)
Neutrino decoupling	1 s	6×10^9	1 MeV	> Weak interaction $e^+ + e^- \leftrightarrow \nu + \bar{\nu}$
Electron-positron annihilation	6 s	2×10^9	500 keV	> $e^+ + e^- \leftrightarrow \gamma + \gamma$
Big Bang nucleosynthesis	3 min	4×10^8	100 keV	> $p, n \rightarrow D, H, He, Li, (Be)$
Matter-radiation equality	60 kyr	3400	0.75 eV	
Recombination	260–380 kyr	1100–1400	0.26–0.33 eV	> $e^- + p^+ \leftrightarrow H + \text{gamma}$
Photon decoupling	380 kyr	1000–1200	0.23–0.28 eV	> CMB
Reionization	100–400 Myr	11–30	2.6–7.0 meV	> $H + \text{gamma} \leftrightarrow e^- + p^+$
Dark energy-matter equality	9 Gyr	0.4	0.33 meV	> Acceleration “start”
Present	13.8 Gyr	0	0.24 meV	> Life

Quantum statistics

$$\cdot (\bar{x}, \bar{p})$$



Grand canonical ensemble

$$f_k = \frac{e^{-(E(N_k) - \mu N_k)/kT}}{Z_{gc}}$$

$$Z_{gc} = \sum_{N_k} e^{-(E(N_k) - \mu N_k)/kT}$$

$$\Phi(T, V, \mu) = -kT \ln(Z_{gc})$$

$$S = -\left. \frac{\delta \Phi}{\delta T} \right|_{V, \mu} \quad P = -\left. \frac{\delta \Phi}{\delta V} \right|_{T, \mu} \quad N = -\left. \frac{\delta \Phi}{\delta \mu} \right|_{T, V} \quad U = \Phi + T \delta_T \Phi + \mu \delta_\mu \Phi \quad \Phi(T, V, \mu) = -PV$$

$$\mu_1 + \mu_2 = \mu_3 + \mu_4 \quad \text{Chemical equilibrium } \mu=0$$

Case: Fermions and Bosons

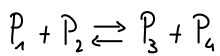
$$z_\alpha = 1 + e^{-(\epsilon_\alpha - \mu)/kT} \quad \text{Fermi-Dirac}$$

$$z_\alpha = (1 - e^{-(\epsilon_\alpha - \mu)/kT})^{-1} \quad \text{Bose-Einstein}$$

$$\Rightarrow n, p, u, s \quad \text{eg. } n_B = \int_0^\infty \frac{g(3)}{\pi^2} \left(\frac{kT}{hc} \right)^3$$

$$S = s \cdot \Delta^3 \propto T^3 \cdot T^{-3} = \text{const.} \quad \text{Entropy is conserved}$$

Reactions



$$T = m \langle \sigma v \rangle$$

$T \gg H$ Thermal equilibrium

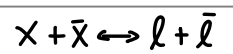
$T \ll H$ Particles decouple / Freeze-out

Freeze-out: Boltzmann eq.

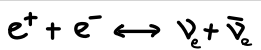
$$\frac{1}{\Delta^3} \frac{d(\dot{M}_\alpha \Delta^3)}{dt} = -\alpha M_1 M_2 + \beta M_3 M_4 = -\langle \sigma v \rangle M_1 M_2 \left[1 - \left(\frac{M_1 M_2}{M_3 M_4} \right)_{eq} \frac{M_3 M_4}{M_1 M_2} \right]$$

$$n = \frac{\#}{V} = \frac{\#}{\Delta^3 V_c} \quad N \approx \Delta^3 n = \frac{\#}{V_c}$$

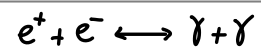
$$(\dot{M}_\alpha \Delta^3) = \Delta^3 (\dot{M}_s + 3H M_s)$$



$$DM \quad \frac{dN_X}{dx} = -\frac{\langle \sigma v \rangle}{\Delta^3 x H} (N_X^2 - N_{Xeq}^2) = -\frac{\lambda}{x^2} (N_X^2 - N_{Xeq}^2) \quad \lambda = \frac{2\pi^2}{45} g_{*s} \frac{M_X^3 \langle \sigma v \rangle}{H(M_X)}$$



CvB

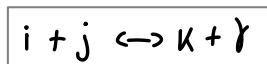


$$\text{reheating: } S_{e^+} + S_{e^-} + S_\gamma = S'_\gamma$$

$$T_\gamma = 2,752 K$$

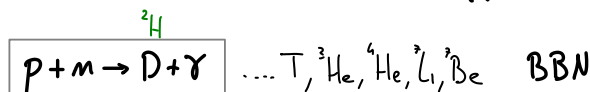
$$T_\nu = 1,95 K$$

Reaction in equilibrium: Saha eq.



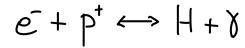
$$\frac{x_i x_j}{x_k} = \frac{g_i g_j}{g_k} \frac{\pi^2}{2 \zeta(3)} \left(\frac{m_e c^2}{kT} \right) e^{-E_i/kT}$$

$$x_i \equiv \frac{n_i}{n_B} \quad \text{ionization fraction}$$



Recombination, Bis

We now consider recombination which is simpler



• From Helmholtz free energy we derive all ensemble properties

$$F(T, V, N) = -kT \ln(Z_c) \quad \text{with} \quad Z_c = \prod_i \sum_{N_i} e^{-(E_i - \mu) N_i / kT} \quad i = e, p, H$$

$$\text{here: } Z_c = \frac{z_e^{N_e} z_p^{N_p} z_H^{N_H}}{N_e! N_p! N_H!}$$

↓

$$\Phi = -kT \ln \left(\frac{z_e^{N_e} z_p^{N_p} z_H^{N_H}}{N_e! N_p! N_H!} \right)$$

$$= -kT \left[\ln(z_e^{N_e} z_p^{N_p} z_H^{N_H}) - \ln(N_e! N_p! N_H!) \right]$$

$$= -kT \left[N_e \ln z_e + \frac{N_p}{N_e} \ln z_p + \frac{N_H}{N_B - N_e} \ln z_H - \ln(N_e!) - \ln(N_p!) - \ln(N_H!) \right]$$

$$\downarrow \quad N_H (\ln N_H - 1) = (N_B - N_e) (\ln N_H - 1)$$

$$-2 \ln(N_e!) \approx -2 N_e (\ln N_e - 1)$$

• Equilibrium at the minimum of $F \Rightarrow \frac{\delta F}{\delta N_e} = 0$ Stirling formula $\lim_{x \rightarrow \infty} \ln(x) = x(\ln(x) - 1)$

$$\frac{\delta F}{\delta N_e} = -kT \left[\ln z_e + \ln z_p - \ln z_H - 2(\ln N_e - 1) + \ln N_H \right] \stackrel{!}{=} 0$$

$$\ln \left(\frac{z_e z_p}{z_H} \right) = \ln \frac{N_e^2}{N_H} = \ln \frac{N_e^2}{N_B - N_e} = \ln \frac{x^2 N_B}{1-x} \Rightarrow \frac{x^2}{1-x} = \frac{z_e z_p}{z_H N_B} \quad x \equiv \frac{n_e}{n_B} \text{ degree of ionization}$$

• $n_B = n \eta$ η ! ratio of baryons and photons and # of photons

• ionization potential of H: $\chi = (m_e + m_p - m_H) c^2 = 13.6 \text{ eV}$

• non relativistic particles: $\varepsilon = m_i c^2 + \frac{p^2}{2m_i} \quad i = e, p, H$ *

• plug z_e, z_p, z_H use continuous distribution (M-B) for the i -th specie

$$z_i = \frac{4\pi g_i V}{(2\pi\hbar)^3} \int_0^\infty p^2 e^{-(\varepsilon - \mu_i)/kT} dp = \frac{g_i V (2\pi m_i kT)^{3/2}}{(2\pi\hbar)^3} e^{-(m_i c^2 - \mu_i)/kT} \quad \mu_e + \mu_p = 0 = \mu_H \text{ at equilibrium}$$

$$\frac{z_e z_p}{z_H} = \frac{g_e g_p}{g_H} V \frac{(2\pi \frac{m_e m_p}{m_H} kT)^{3/2}}{(2\pi\hbar)^3} e^{-\frac{(m_e + m_p - m_H) c^2}{kT}} \quad \frac{g_H}{g_e g_p} = \frac{4}{2 \cdot 2} \quad g_H = 1 + 3 \quad \begin{matrix} \uparrow \\ \text{right} \end{matrix} \quad \begin{matrix} \uparrow \\ \text{triplets} \end{matrix}$$

$$\Rightarrow \frac{x^2}{1-x} = \frac{(2\pi m_e kT)^{3/2}}{(2\pi\hbar)^3 n_B} e^{-\chi/kT} \\ = \frac{\sqrt{\pi}}{4\sqrt{2} \zeta(3) \eta} \left(\frac{m_e c^2}{kT} \right)^{3/2} e^{-\chi/kT}$$

$$\frac{N_B}{V} \quad n_B = n \eta = 2 \eta \frac{\zeta(3)}{\pi^2} \left(\frac{kT}{\hbar c} \right)^3$$

Saha's equation

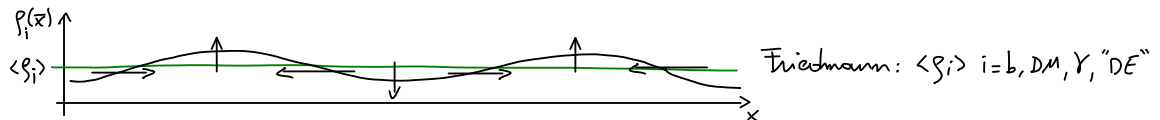
η tunes the process!

Part IV

Cosmic structure formation

Cosmic structure formation

- The universe is not perfectly homogeneous
- Initial energy-density fluctuations produced by inflation (seeds)
- Sound waves in the medium for $p \neq 0$ (harmonic oscillations)
- Gravity and cosmic expansion cause them to evolve



- Formation of filaments and voids on scales of $\sim 50 \text{ Mpc}$ (very large but still $\ll R_H$)
 - " " galaxy clusters " " $\sim 1 \text{ Mpc}$
 - " " galaxies " " $\sim 0,1 \text{ Mpc}$
- Treat the components as fluids $\Rightarrow$ use hydrodynamics
 - small overdensities: Weak field limit
 - scales $\ll R_H$: ignore retardation and curvature $\Rightarrow$ Newtonian gravity suffice
- Use perturbative approach when in linear regime: $\rho(\vec{x}) = \langle \rho \rangle + \delta \rho$ $\delta \rho \ll \langle \rho \rangle$
- Spherical collapse and n -body simulation when in non linear regime

• Cold Dark Matter (CDM)

Its existence and characteristics are necessary to explain observations

- $\hookrightarrow$ cold : non relativistic before decoupling
- $\hookrightarrow$ non baryonic : "exotic" particle
- $\hookrightarrow$ non collisional : zero pressure $P=0$
- $\hookrightarrow$ dark : only gravitational interaction (no emissions/absorptions) $\Rightarrow$ does not radiate away!

• Baryons $\rightarrow$ radiates $\Rightarrow$ further collapse

$\rightarrow$ galaxies / stars form later

$\rightarrow$ complex physics: feedback, Super-Novae, AGNs,...

• Neutrinos $\rightarrow$ remains relativistic for longer time (because of small mass)

$\rightarrow$ their diffusion affects structure formation

Length scales and structure formation

- Relevant scales:
- | | | |
|-------------------------|-------------------------------|-------------------|
| 1) Cosmological horizon | $R_H(t) = a(t) \int_0^t c dt$ | causal connection |
| 2) Jeans scale | $\lambda_J = R_J$ | collapse yes/no |
| 3) Diffusion scale | λ_D | diffusion |

2) Jeans scale: Criteria for gravitational collapse

- R_J sets a criteria for having a gravitational collapse for $R > R_J$ (Jeans criteria)
- it is related to the balance of the system

- Spherical region: $R, \rho, M = \rho \frac{4\pi}{3} R^3, v = \text{typical values from Maxwell-Boltzmann distrib.}$

- Energy balance: $\begin{cases} E_k \approx \frac{1}{2} M v^2 \\ E_g \approx -\frac{G M^2}{R} \end{cases} \Rightarrow E_k + \frac{1}{2} V = 0 \quad \text{virial theorem}$

$$\frac{1}{2} M v^2 = \frac{1}{2} \frac{G M^2}{R} \Rightarrow \frac{G}{R} \rho R^3 \Rightarrow R_J = v \sqrt{\frac{1}{G \rho}}$$

- Force balance: $\begin{cases} F_p \approx \frac{P \cdot A}{M} = \frac{P \cdot R^2}{M} = \frac{P R^2}{\rho R^3} = \frac{c_s^2}{R} \\ F_g \approx -\frac{G M}{R^2} \end{cases} \Rightarrow \frac{c_s^2}{R} = \frac{G M}{R^2} = \frac{G \rho R}{R} \Rightarrow R_J = c_s \sqrt{\frac{1}{G \rho}}$

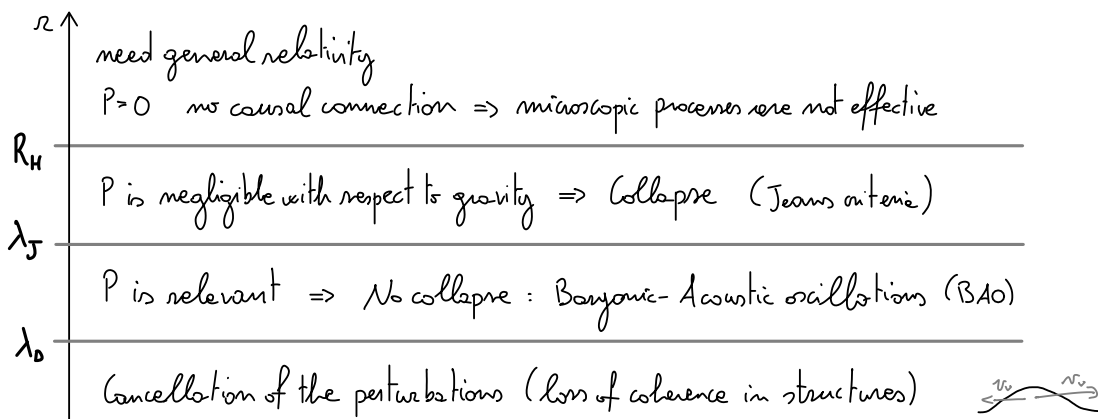
(per unit of mass)

- Time scales: $\begin{cases} \tau_{ff} \sim (G \rho)^{-1/2} & \text{free fall time} \\ \tau_{cool} \sim \frac{2R}{c_s} & \text{cooling time} \end{cases} \Rightarrow \frac{1}{2} (G \rho)^{-1/2} = \frac{R}{c_s} \Rightarrow R_J = c_s \sqrt{\frac{1}{G \rho}}$

▫ Jeans mass: $M_J = \frac{4\pi}{3} R_J^3 \rho$ mass within R_J

3) Diffusion scale λ_D : sets a scale "smoothing" the overdensities

depends on the velocity of particles



The density contrast and a model for gravitational collapse

- For all relativistic species and/or $\lambda > R_H$ you need GR
- We have a simple general GR solution valid for $\lambda > \lambda_3$ and $\lambda > R_H$:
 - treat overdensity as a "closed universe" immersed in a flat one (background) $\Rightarrow$ use Friedmann eq.

$$H^2 = \frac{8\pi G}{3} \rho - \frac{K}{a^2} \quad \begin{cases} H_B = \frac{8\pi G}{3} \rho_B \quad (K=0) \\ H_1^2 = \frac{8\pi G}{3} \rho_1 - \frac{c^2}{a^2} \quad (K=1) \end{cases} \quad \boxed{H_B = H_1} \quad \frac{8\pi G}{3} \rho_B \left(\frac{\rho_1 - \rho_B}{\rho_B} \right) = \frac{c^2}{a^2}$$

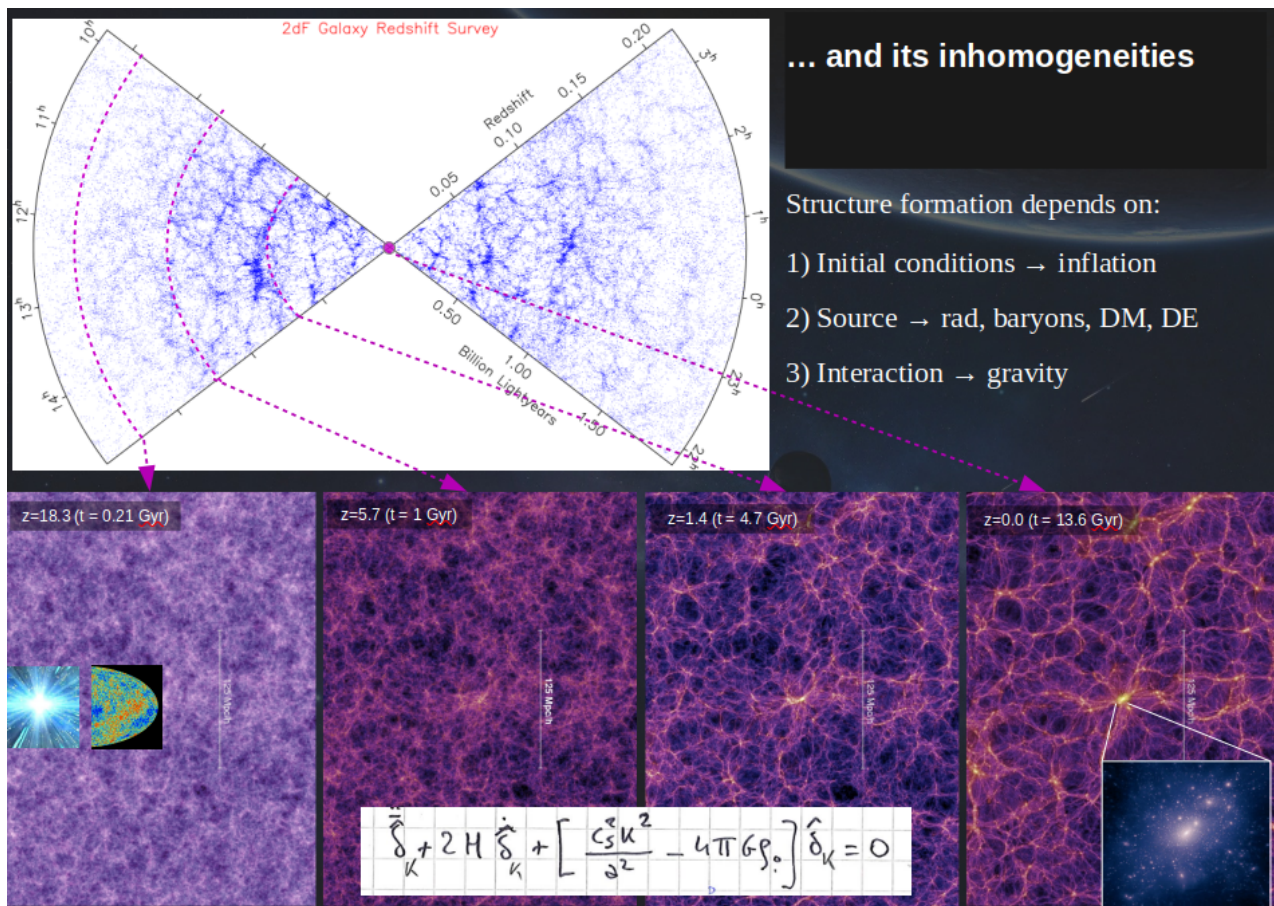
$\delta(a) = \text{density contrast}$

$$\Rightarrow \boxed{\delta(a)} = \frac{3c^2}{8\pi G \rho_B(a) a^2} \propto a^{3+3w-2} \propto \boxed{a^{1+3w}} \propto \begin{cases} a^2 a^t & \text{for } w = \frac{1}{3} \text{ Radiation domination era} \\ a^2 a^{t^{1/3}} & \text{for } w = 0 \text{ Matter domination era} \end{cases}$$

$$\rho = \rho_0 \left(\frac{a}{a_0} \right)^{3(1+w)} \quad \text{polytropic eq. of state}$$

↑
the non dominant components follow the dominant one

- $(\delta) \leftrightarrow (a) \leftrightarrow (t) \Rightarrow \delta$ can be used as a time variable
growth factor, more later...



Structure formation in the linear regime

Assumptions

- $\Lambda \leftrightarrow$ DE enters in the dynamic of the universe, they do not participate to gravit. collapse
- Focus on scales $\lambda < R_H$: neglect $\left\{ \begin{array}{l} \text{curvature} \\ \text{finite speed of light (retardation)} \end{array} \right.$
we deal with non relativistic velocities
use: Newtonian gravity / dynamic
- Many neutral particles $\rightarrow$ treat as fluid $\rightarrow$ macroscopic quantities
Properties: density, velocity, pressure, entropy, self gravity

$$\begin{array}{ccccc} \rho(\epsilon, \vec{r}) & \vec{v}(\epsilon, \vec{r}) & P(\epsilon, \vec{r}) & S & \varphi(\epsilon, \vec{r}) \\ (1) & (3) & (1) & (1) & (1) \end{array} \quad 7 \text{ variables}$$

Hydrodynamic

Ideal Fluid: $T_{\mu\nu} = \left(\rho + \frac{P}{c}\right) u_\mu u_\nu + P \eta_{\mu\nu}$

Energy-momentum conservation $\sum_\mu T^{\mu\nu} = 0$ eq. of motion of fluids $\left. \vphantom{\sum_\mu T^{\mu\nu} = 0} \right\}$ Non relativistic limit

(1) energy conservation ($\nu=0$)	$\delta_\epsilon \xi + \vec{\nabla}(\xi \vec{v}) + P \vec{\nabla} \vec{v} = 0 \quad \oplus$	$\vec{\nabla} = (\delta_x, \delta_y, \delta_z)^T$
(2) momentum conservation ($\nu=1,2,3$)	$\frac{d\vec{v}}{dt} = \frac{\delta \vec{v}}{\delta t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\frac{\vec{\nabla} P}{\rho} - \vec{\nabla} \varphi$ (Euler eq.)	
(3) matter conservation	$\frac{\delta \rho}{\delta t} + \vec{\nabla}(\rho \vec{v}) = 0$ (Continuity equation)	
(4) equation of state	$P(\rho, S)$	
entropy evolution	$\oplus \frac{\delta S}{\delta t} = 0$ (adiabatic)	$P(\rho) = w c^2 \rho \quad 0 \leq w < 1$
(5) eq. of motion of gravity	$\vec{\nabla}^2 \varphi = 4\pi G \rho$ (Poisson eq.)	

$\Leftrightarrow$ Closed system of equations $\Rightarrow$ 7 variables and 7 equations

$$\oplus \delta_\epsilon \xi + \vec{\nabla}(\xi \vec{v}) + P \vec{\nabla} \vec{v} = \rho T \frac{d\tilde{S}}{dt} = 0 \quad \tilde{S} \equiv \frac{S}{\rho} \text{ specific entropy}$$

$\uparrow$
1st law of thermodynamics $d\xi + P dV = T ds$

(1) Linearize equations for small perturbations

$$\begin{cases} \rho = \rho_B + \delta\rho = \rho_B(1+\delta) & \delta(\epsilon, \vec{x}) \equiv \frac{\rho(\epsilon, \vec{x}) - \rho_B(\epsilon, \vec{x})}{\rho_B(\epsilon, \vec{x})} = \frac{\delta\rho}{\rho_B} \ll 1 & |\delta\vec{v}| \ll |\vec{v}_B| & \delta P \ll P_B & |\delta\varphi| \ll 1 \\ \vec{v} = \vec{v}_B + \delta\vec{v} \\ P = P_B + \delta P \\ \varphi = \varphi_B + \delta\varphi \end{cases}$$

(2) Meaning of the velocity perturbation $\vec{x} = \text{physical coordinates}$ $\vec{x} \equiv \text{comoving}$ $\vec{u} \equiv \frac{\delta\vec{v}}{a}$ comoving proper velocity

$\vec{x} = a \vec{\bar{x}}$

$\vec{v} = \frac{d\vec{x}}{dt} = \dot{a}\vec{\bar{x}} + a\dot{\vec{\bar{x}}} = H a \vec{\bar{x}} + a \dot{\vec{\bar{x}}}$

 $\vec{\nabla} \cdot \vec{v}_B \neq 0$ "radial" $H \vec{x} = \vec{v}_B$ Hubble flow $\delta\vec{v} = a \dot{\vec{\bar{x}}}$ Proper motion(3) "Transform" time derivatives accounting for comoving coordinates

$\bar{\nabla} = \bar{\nabla}_{\vec{x}} = \frac{\partial}{\partial \delta \vec{x}} = \frac{1}{a} \bar{\nabla}_{\vec{x}}$

$$df(\epsilon, \vec{x}) = \delta_\epsilon f d\epsilon + \bar{\nabla} f d\vec{x} = \delta_\epsilon f d\epsilon + a \bar{\nabla} f d\vec{\bar{x}} + \vec{\bar{x}} a \bar{\nabla} f \frac{da}{a} = (\delta_\epsilon + \vec{\bar{x}} H \bar{\nabla}_{\vec{x}}) f d\epsilon + \bar{\nabla}_{\vec{x}} f d\vec{\bar{x}}$$

a) Continuity eq.

$$\begin{aligned} \delta_\epsilon \rho + \bar{\nabla}(\rho \vec{v}) &= \delta_\epsilon(\rho_B + \delta\rho) + \bar{\nabla}[(\rho_B + \delta\rho)(\vec{v}_B + \delta\vec{v})] = \delta_\epsilon \rho_B + \bar{\nabla}(\rho_B \vec{v}_B) + \delta_\epsilon \delta\rho + \bar{\nabla}(\rho_B \delta\vec{v} + \delta\rho \vec{v}_B + \delta\rho \delta\vec{v}) = 0 \\ &\approx \delta_\epsilon \rho_B + \rho_B \bar{\nabla} \delta\vec{v} + \vec{v}_B \bar{\nabla} \delta\rho + \delta\rho \bar{\nabla} \vec{v}_B = 0 \quad \leftarrow \delta_\epsilon \rho = \rho_B \delta_\epsilon : \delta_\epsilon \delta\rho = \rho_B \delta_\epsilon \delta + \delta \delta_\epsilon \rho_B = \rho_B \delta_\epsilon \delta - \delta \cdot \rho_B \bar{\nabla} \vec{v}_B \\ &= \rho_B \delta_\epsilon \delta - \delta \rho_B \bar{\nabla} \vec{v}_B + \rho_B \bar{\nabla} \delta\vec{v} + \rho_B H \vec{x} a \bar{\nabla} \delta + \delta \rho_B \bar{\nabla} \vec{v}_B = 0 \quad \leftarrow \bar{\nabla} \delta\rho = \rho_B \bar{\nabla} \delta \quad \leftarrow \bar{\nabla}(\rho_B \vec{v}_B) \text{ continuity} \\ &\Rightarrow \boxed{\dot{\delta} + \bar{\nabla}_{\vec{x}} \vec{u} = 0} \end{aligned}$$

b) Euler eq.

$$\begin{aligned} (\delta_\epsilon + \vec{v} \cdot \bar{\nabla}) \vec{v} &= -\frac{\bar{\nabla} P}{\rho} - \bar{\nabla} \varphi \quad \left[\delta_\epsilon + (\vec{v}_B + \delta\vec{v}) \cdot \bar{\nabla} \right] (\vec{v}_B + \delta\vec{v}) = -\frac{\bar{\nabla}(P_B + \delta P)}{(\rho_B + \delta\rho)} - \bar{\nabla}(\varphi_B + \delta\varphi) \\ \delta_\epsilon \vec{v}_B + \delta_\epsilon \delta\vec{v} + \vec{v}_B \bar{\nabla} \delta\vec{v} + \delta\vec{v} \bar{\nabla} \vec{v}_B &= -\frac{\bar{\nabla} P_B}{\rho_B} - \frac{\bar{\nabla} \delta P}{\rho_B} - \bar{\nabla} \varphi_B - \bar{\nabla} \delta\varphi \quad \text{as before, background follows Euler eq.} \\ \delta_\epsilon \delta\vec{v} + (\vec{v}_B \cdot \bar{\nabla}) \delta\vec{v} + (\delta\vec{v} \cdot \bar{\nabla}) \vec{v}_B &= -\frac{\bar{\nabla} \delta P}{\rho_B} - \bar{\nabla} \delta\varphi \quad \leftarrow \begin{aligned} &1) (\delta\vec{v} \cdot \bar{\nabla}) \vec{v}_B = \delta v^i \nabla_i (H \vec{x}) = \delta v^i H \frac{\partial x^j}{\partial x^i} = \delta v^i H = a \dot{u}^i H \\ &2) (\vec{v}_B \cdot \bar{\nabla}) \delta\vec{v} = (H a \vec{x} \cdot \bar{\nabla}) \delta\vec{v} = (H \vec{x} \cdot \bar{\nabla}_{\vec{x}}) \delta\vec{v} \\ &3) \dot{\delta\vec{v}} = (a \dot{\vec{u}}) = \dot{a} \vec{u} + a \dot{\vec{u}} = a H \vec{u} + a \dot{\vec{u}} \\ &4) \frac{\delta P}{\rho_B} = c_s^2 \quad (\text{eq. of state}) \quad \delta\rho = \rho_B \delta \end{aligned} \\ \boxed{\dot{\vec{u}} + 2H \vec{u} = -c_s^2 \frac{\bar{\nabla} \delta}{\rho_B a^2} - \bar{\nabla}_{\vec{x}} \frac{\delta\varphi}{a^2}} \end{aligned}$$

c) Poisson eq. $\bar{\nabla}^2 \varphi = 4\pi G \rho \quad \bar{\nabla}^2(\varphi_B + \delta\varphi) = 4\pi G(\rho_B + \delta\rho) \quad \boxed{\bar{\nabla}_{\vec{x}}^2 \delta\varphi = 4\pi G a^2 \rho_B \delta}$

• From now on, $\bar{\nabla} \equiv \bar{\nabla}_x$

$\dot{\delta} + \bar{\nabla} \bar{u} = 0$	Continuity eq.
$\ddot{u} + 2H\dot{u} = -c_s^2 \frac{\bar{\nabla}^2 \delta}{\bar{a}^2} - \bar{\nabla} \frac{\delta \phi}{\bar{a}^2}$	Euler eq.
$\bar{\nabla}^2 \delta \phi = 4\pi G \bar{a}^2 \rho_b \delta$	Poisson eq.
$\frac{\delta P}{\delta \rho} = c_s^2$	eq. of state $P = P(\rho)$

6 equations for 6 variables: $\delta, \bar{u}, P, \delta \phi$

closed system of eq.

↓

Solve it

$\delta_e(\text{continuity}) : \dot{\delta} + \bar{\nabla} \bar{u} = 0$ ↗ Poisson

$\bar{\nabla}(\text{Euler}) : \bar{\nabla} \ddot{u} + 2H \bar{\nabla} \dot{u} = -c_s^2 \frac{\bar{\nabla}^2 \delta}{\bar{a}^2} - \frac{\bar{\nabla}^2 \delta \phi}{\bar{a}^2} = -c_s^2 \frac{\bar{\nabla}^2 \delta \rho}{\bar{\rho}_b \bar{a}^2} - 4\pi G \rho_b \delta$ ↖ continuity

$$\ddot{\delta} + 2H\dot{\delta} = c_s^2 \frac{\bar{\nabla}^2 \delta}{\bar{a}^2} + 4\pi G \rho_b \delta$$

Density perturbation eq.

- H acts as a damping term: expansion affects structure formation!
- Gravity and Pressure are the source term
they depend on the fluid, e.g. DM is pressureless

• Decompose in plane waves i.e. go to Fourier space

$$\delta(t, \vec{x}) = \int \frac{d^3 k}{(2\pi)^3} \hat{\delta}(t, \vec{k}) e^{-i\vec{k}\vec{x}}$$

Ansatz for the solution

$\hat{\delta}_k$ Fourier mode, spatial derivatives become $\bar{\nabla} \rightarrow -i\vec{k}$

$$\bar{\nabla} \delta(t, \vec{x}) = \int \frac{d^3 k}{(2\pi)^3} (-i\vec{k}) \hat{\delta}(t, \vec{k}) e^{-i\vec{k}\vec{x}}$$

$$\ddot{\hat{\delta}}_k + 2H\dot{\hat{\delta}}_k = \left(-\frac{c_s^2 k^2}{\bar{a}^2} + 4\pi G \rho_b \right) \hat{\delta}_k$$

For matter components

$$\ddot{\hat{\delta}}_k + 2H\dot{\hat{\delta}}_k = \left(-\frac{c_s^2 k^2}{3\bar{a}^2} + \frac{32\pi}{3} G \rho_b \right) \hat{\delta}_k$$

For relativistic fluid

here, we used: $P = \frac{1}{3} \rho c^2 \Rightarrow c_s^2 \equiv \frac{P}{\rho} \quad c_s = \frac{c}{\sqrt{3}}$ (radiation domination)

$\bar{\nabla}(\rho \bar{u}) \rightarrow \bar{\nabla}[(\rho + \frac{P}{c^2}) \bar{u}] = \frac{4}{3} \bar{\nabla}(\rho \bar{u})$ in continuity eq.

$\rho \rightarrow 2\rho$ in Poisson eq.

• in the linear regime, $\hat{\delta}_k$ modes are decoupled:

⇒ They evolve independently from each other

⇒ No power transfer to $\neq$ modes

i.e. shape of power spectrum will not change $P(k) \propto k^{n_s}$ $n_s = 0, 96$

Simple case: static universe: $H=0$ • Matter domination

$$\ddot{\delta}_k + \omega_k \dot{\delta}_k = 0 \quad \omega_k = \sqrt{\frac{c_s^2 k^2}{a^2} - 4\pi G \rho_b} \quad \text{oscillator} \rightarrow \text{solution } \hat{\delta}_k \propto e^{i\omega_k t}$$

$\omega_k \in \mathbb{R}$: if $\frac{c_s^2 k^2}{a^2} \geq 4\pi G \rho_b \Rightarrow k \geq \frac{a}{c_s} \sqrt{4\pi G \rho_b} \equiv k_J \quad \lambda_J = \frac{2\pi}{k_J} = \frac{c_s}{a} \sqrt{\frac{4\pi a^2}{G \rho_b}}$ Jeans scale

$\lambda < \lambda_J$ harmonic oscillations (No collapse!)

$\omega_k \in \mathbb{C}$: $\lambda > \lambda_J$ growth/decay
with $\omega_k = a + ib$ $b \neq 0 \quad \hat{\delta}_k \propto e^{ia} \cdot e^b \quad b > 0$ damped oscillations

• Radiation domination

$$\omega_k = \sqrt{-\frac{c_s^2 k^2}{3a^2} + \frac{32\pi G \rho_b}{3}} \stackrel{!}{=} 0 \quad \Rightarrow \quad k_J = \frac{a}{c_s} \sqrt{32\pi G \rho_b}$$

$$\lambda_J = \frac{2\pi}{k_J} = \frac{c_s}{a} \sqrt{\frac{4\pi a^2}{32\pi G \rho_b}} = \frac{c_s}{2a} \sqrt{\frac{\pi}{2G \rho_b}} = \frac{c_s}{2a} \sqrt{\frac{\pi \cdot 2\pi a^2}{2 \cdot 3 H^2}} = \frac{\pi}{\sqrt{3}} \frac{c_s}{H} \approx R_H$$

$\Rightarrow$ growth is forbidden within the Hubble horizon!

$\rightarrow$ as baryons are coupled to radiation, their fluctuations will not grow either!

Photons: $w = \frac{1}{3} \Rightarrow c_s^2 = \frac{P}{\rho} = \frac{1}{3} c^2 \quad \lambda_J \approx R_H \Rightarrow$ no growth within the horizon

Baryons: $c_s > 0 \Rightarrow (\lambda > \lambda_J)$ growth or $(\lambda < \lambda_J)$ Baryonic Acoustic Oscillations (BAO)

before decoupling with radiations $c_s \sim c \Rightarrow$ baryons perturbations are damped out

after " " " $c_s \ll c \Rightarrow$ they can fall in DM potential well that could grow because not coupled to photons

Dark Matter: $w = 0 \Rightarrow P = 0 \Rightarrow c_s^2 = \frac{P}{\rho} = 0 \Rightarrow \lambda_J = 0$ growth on all scales

$\Rightarrow c_s$ has no meaning... but: $\lambda_J = \langle v^2 \rangle^{-1/2} \sqrt{\frac{\pi}{G \rho_b}}$

You will get virialized structures: Dark Matter Haloes, later....

- Cold Dark Matter (CDM): $w \sim 0 \Rightarrow \lambda_D = 0 \Rightarrow$ "all" scales can collapse

- Warm " " (WDM)

- Hot Dark Matter (HDM) w larger (relativistic) $\lambda_D > 0 \Rightarrow$ the spectrum is truncated ("smaller" structures can not form)

WDM and HDM are ruled out by observations

• The Growth Factor: D

◦ Assume

$\lambda \gg \lambda_J \Rightarrow c_s = 0$ (pressureless fluid)

for λ large, pressure is negligible

$\Omega = 1$ (flat geometry) and $\Omega_\Lambda = 0$ ("early times") $\Rightarrow H^2 = \frac{8\pi G}{3} \rho_B$ Friedman eq.

◦ Apply assumptions

$$M: \ddot{\delta}_k + 2H\dot{\delta}_k = \left(-\frac{c_s^2 k^2}{\Omega^2} + 4\pi G \rho_B \right) \hat{\delta}_k \quad \ddot{\delta}_k + 2H\dot{\delta}_k = \frac{3}{2} H^2 \hat{\delta}_k \quad (1)$$

$$R: \ddot{\delta}_k + 2H\dot{\delta}_k = \frac{3}{2} \frac{H^2}{\Omega} \hat{\delta}_k \quad \ddot{\delta}_k + 2H\dot{\delta}_k = 4 H^2 \hat{\delta}_k$$

◦ Solve eq.

• Matter domination: $\rho_B = \rho_{\text{matter}} \quad H = \frac{2}{3t} \quad \text{ansatz } \hat{\delta} \propto t^m$

$$m(m-1)t^{m-2} + 2 \cdot \frac{2}{3t} m t^{m-1} = \frac{3}{2} \frac{4}{9t^2} t^m \quad m^2 + \frac{1}{3}m - \frac{2}{3} = 0 \quad m_{\pm} = -1, \frac{2}{3}$$

• Radiation domination: $\rho_B = \rho_R \quad H = \frac{1}{2t}$

$$m(m-1)t^{m-2} + 2 \cdot \frac{1}{2t} m t^{m-1} = \frac{4}{4t^2} t^m \quad m^2 - 1 = 0 \quad m_{\pm} = 1, -1$$

$$\Rightarrow \begin{cases} \hat{\delta}_k \propto \begin{pmatrix} 2 \\ \Omega^{-3/2} \end{pmatrix} & \text{Matter domination era} \\ \hat{\delta}_k \propto \begin{pmatrix} 2 \\ \Omega^{-1} \end{pmatrix} & \text{Radiation " " } \end{cases} \quad \text{decaying modes are irrelevant}$$

◦ For $\Omega \neq 1$ and $\Omega_\Lambda > 0$

slower evolution at late time

$$\delta(\varphi) = \delta_0 D_+(\varphi) \quad D_+(\varphi) = \frac{5\varphi}{2} \Omega_m \left[\Omega_m^{4/3} - \Omega_\Lambda + \left(1 + \frac{\Omega_m}{2}\right) \left(1 + \frac{\Omega_\Lambda}{\Omega_0}\right) \right] \quad \text{Excellent approximation}$$

Velocity perturbations

Back to the linearized equations (matter domination)

$$\dot{\delta} + \nabla \cdot \bar{u} = 0$$

Continuity eq.

$$\dot{\bar{u}} + 2H\bar{u} = -\frac{1}{a^2} \nabla \delta - \nabla \frac{\delta\varphi}{a^2}$$

Euler eq.

← Neglect pressure, i.e. consider large scales

$$\nabla^2 \delta\varphi = 4\pi G a^2 \bar{\rho} \delta$$

Poisson eq.

A numerical solution is needed but we can make 2 simple considerations:

1) velocity fields point in the direction given by $-\nabla\varphi$

2) amplitude $u(t)$ is time dependent only (the growth factor is $D(t)$)

$$\Rightarrow \bar{u}(\bar{x}, t) = -u(t) \nabla \delta\varphi$$

$$\begin{aligned} \nabla \bar{u}(\bar{x}, t) &= -u(t) \nabla^2 \delta\varphi \\ &= -u(t) 4\pi G a^2 \bar{\rho} \delta \end{aligned} \quad \text{Poisson eq.}$$

$$\begin{aligned} H \frac{\delta\delta}{\delta \ln a} &= u(t) 4\pi G a^2 \bar{\rho} \delta \\ &= u(t) \frac{3}{2} H^2 \Omega_m a^2 \delta \end{aligned}$$

$$\text{Continuity eq. } \nabla \bar{u} = -\dot{\delta} = -\frac{\delta\delta}{\delta a} \dot{a} = -H \frac{\delta\delta}{\delta \ln a}$$

$$4\pi G \bar{\rho} = 4\pi G \frac{3H^2}{2} \Omega_m = \frac{3}{2} H^2 \Omega_m$$

$$\text{growth factor } \delta = \delta_0 D_+(a)$$

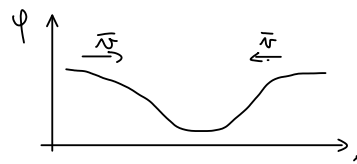
$$\frac{\delta \ln D_+}{\delta \ln a} = \frac{3}{2} H \Omega_m a^2 \underline{u(t)}$$

$$u(t) = \frac{2f(\Omega_m)}{3H\Omega_m a^2}$$

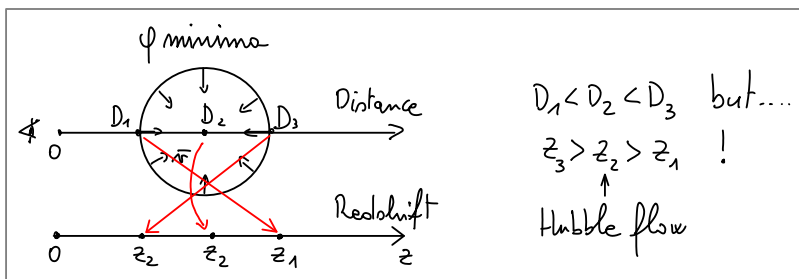
$$f(\Omega_m) = \frac{\delta \ln D_+}{\delta \ln a} \approx \Omega_m^{0.6} \quad \text{very good approximation}$$

$$\Rightarrow \bar{u}(\bar{x}, t) = -\frac{2f(\Omega_m)}{3H\Omega_m a^2} \nabla \delta\varphi$$

$$\delta \bar{v} = a \bar{u}$$

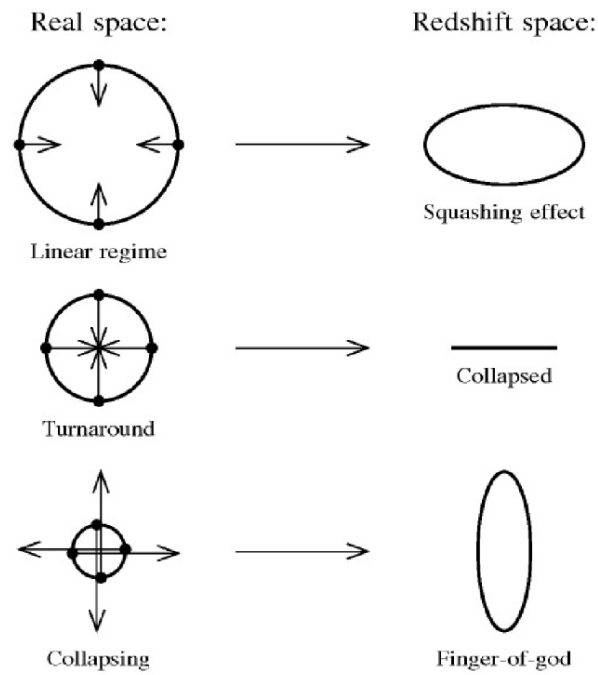


in-flow toward potential minima

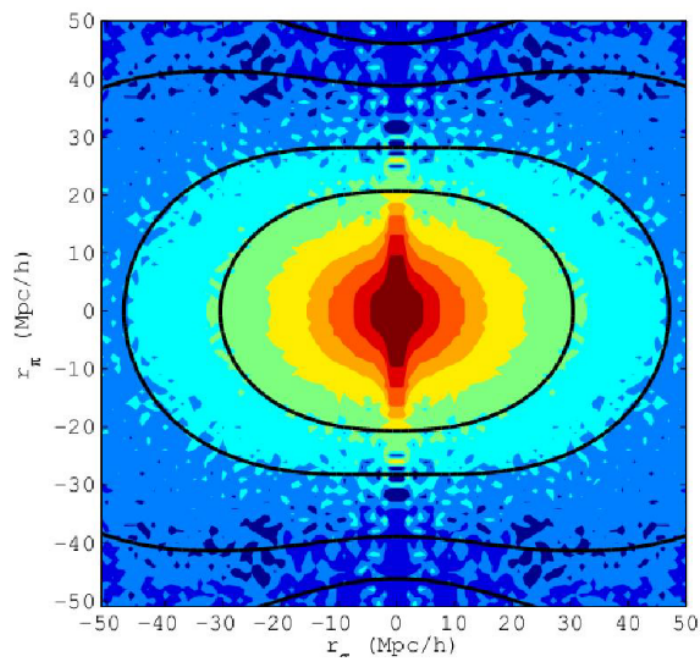


Redshift space distortion
Fingers of God
in the non-linear regime

• Vortical solutions with $\nabla \cdot \bar{u} = 0$ are $\bar{u} = \nabla \times \bar{\omega}$ occurs only where $\delta = 0$



A. J. S. Hamilton, astro-ph/9708102

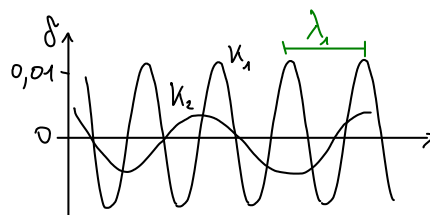


Reid et al. 2012

Quantifying the density fluctuations

- Recall :

- Field $\mathcal{S} = \frac{\mathcal{S} - \mathcal{S}_B}{\mathcal{S}_B}$ $\langle \mathcal{S} \rangle = 0$ by construction
decomposed in k -modes $\mathcal{S}_k$



Total S field is superposition of $\hat{J}_k$ moles

In the linear regime, $\hat{\mathcal{I}}_n$ modes are decoupled

• $\delta, v, \dots$ are random variables, therefore:

- we know how they evolve

eg. $\delta(\vec{x}, \vec{x}) = \delta_0(\vec{x}) \cdot a \Rightarrow$ if $\delta_0 = 0$ at a contain $\vec{x}$ it will remain 0
 matter domination $\nearrow$
 linear regime
 $\delta_0 > 0$ " " " " it will grow
 $\delta_0 < 0$ " " " " " " decrease

valid until $|\delta| \sim 1$
(linear regime)

- but we do not know the value in a certain position $\bar{x}$

- Need for a statistical approach

do not study $\delta(x, y, z)$, study moments: mean, variance, skewness, ...

- mean is not interesting: $\langle \delta \rangle = 0$ by construction

- variance as a function of separation $|\vec{r}|$ $\xi(r) \equiv \langle \delta(\vec{x}) \delta(\vec{x} + \vec{r}) \rangle_x$ correlation function

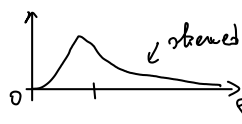
" " " " " mode $|\bar{k}|$ $P(k)$

Power spectrum

- higher momentums are necessary only if the field is not Gaussian

- As the field evolves it will become non-Gaussian because $\varphi \in [0, \infty[= [\text{"voids"}, \text{"galaxies"}]$

$\mathcal{G}$ can become $\emptyset$ (void), $\mathcal{G}$ can grow arbitrarily



$$\left(\begin{array}{ll} \hat{\mathcal{S}}(\epsilon, \bar{\kappa}) = \int_{-\infty}^{\infty} d^3x \mathcal{S}(\epsilon, \bar{x}) e^{i\bar{\kappa}\bar{x}} & \text{Fourier transform} \\ \mathcal{S}(\epsilon, \bar{x}) = \int_{-\infty}^{+\infty} \frac{d^3\kappa}{(2\pi)^3} \hat{\mathcal{S}}(\bar{\kappa}, \epsilon) e^{-i\bar{\kappa}\bar{x}} & \text{back Fourier transform} \end{array} \right)$$

Correlation function and Power spectrum

Correlation function

$$\begin{aligned}
 \xi(x, t) &\equiv \langle \delta^*(\vec{x}, t) \delta(\vec{x} + \vec{r}, t) \rangle = \left\langle \int \frac{d^3k}{(2\pi)^3} \hat{\delta}^*(\vec{k}, t) e^{i\vec{k}\vec{x}} \cdot \int \frac{d^3k'}{(2\pi)^3} \hat{\delta}(\vec{k}', t) e^{-i\vec{k}'(\vec{x} + \vec{r})} \right\rangle \\
 &\stackrel{\delta = \delta^* \text{ because } \delta \in \mathbb{R}}{=} \frac{1}{(2\pi)^6} \int d^3k \int d^3k' \langle \hat{\delta}^*(\vec{k}, t) \hat{\delta}(\vec{k}', t) \rangle e^{i(\vec{k}\vec{x} - \vec{k}'\vec{x} - \vec{k}'\vec{r})} \\
 &= \frac{1}{(2\pi)^3} \int d^3k \int d^3k' (2\pi)^3 P(\vec{k}, t) \delta_p(\vec{k} - \vec{k}') e^{i(\vec{k}\vec{x} - \vec{k}'\vec{x} - \vec{k}'\vec{r})} \\
 &= \frac{1}{(2\pi)^3} \int d^3k P(\vec{k}, t) e^{-i\vec{k}\vec{r}} \quad \text{Wiener-Kitching Theorem ("Parseval")}
 \end{aligned}$$

Power spectrum

$$\langle \hat{\delta}^*(\vec{k}, t) \hat{\delta}(\vec{k}', t) \rangle \equiv (2\pi)^3 P(k, t) \delta_p(\vec{k} - \vec{k}')$$

$\uparrow$ uncorrelated modes
 $\uparrow$ power of each mode, $k = |\vec{k}|$ not $\vec{k}$ isotropy
 $\uparrow$ expectation value over all volumes (ensembles)
 ergodic principle: $\neq$ volumes are statistically equivalent

A convenient expression: $\xi(r) \stackrel{\text{isotropy}}{=} 2\pi \int \frac{k^2 dk}{(2\pi)^3} P(k) \int_0^\pi \sin\theta d\theta e^{-ikr\cos\theta} = 4\pi \int \frac{k^2 dk}{(2\pi)^3} P(k) \frac{\sin(kr)}{kr}$

The way around: $P(k) = \int d^3x \xi(x) e^{i\vec{k}\vec{x}} = 2\pi \int \xi(x) x^2 dx \int_0^\pi e^{ikx\cos\theta} \sin\theta d\theta = 4\pi \int_0^\infty x^2 \frac{\sin(kx)}{kx} \xi(x) dx$
 $\uparrow$ isotropy $\Rightarrow$ spherical coord.

The correlation function is the back Fourier transform of the Power spectrum

" Power spectrum " " Fourier transform " " correlation function

Probability distribution of the modes

$p_k(\hat{\delta})$ = probability to have a fluctuation $\hat{\delta}_k$ in a $d\hat{\delta}$ bin

$$P(\vec{k}, t) = \frac{1}{(2\pi)^3} \langle \hat{\delta}^2(\vec{k}, t) \rangle = \int_{-\infty}^{\infty} d\hat{\delta} p_k(\hat{\delta}) \hat{\delta}^2(\vec{k}, t)$$

if p_k is Gaussian $\Rightarrow$ $P(k)$ stores all information if Gaussianity
 No need of tri-spectrum $\langle \delta(k) \delta(k') \delta(k'') \rangle = 0$
 " " " quadri....

Normalization of the power spectrum and σ_8

• $\sigma_8 \equiv \int_{-\infty}^{\infty} \frac{dk^3}{(2\pi)^3} P(k) \hat{W}_R^2(k)$ window function $\hat{W}$:

- normalized such that: $\int d^3x W(\vec{x}) \equiv 1$
- "selects/weights" certain modes
- used to have a finite variance because $\int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3} P(k) = \infty$, e.g. $P(k) \propto k^q$ ($P(k) > 0$)

$= 4\pi \int_0^{\infty} \frac{k^2 dk}{(2\pi)^3} P(k) W_R^2(k)$

• variance of a convolved field $\delta_c(\vec{x})$, in fact:

$$\delta_c(\vec{x}, t) = \int \delta(\vec{x}, t) W_R(|\vec{x} - \vec{x}'|) d^3x' \quad \longleftrightarrow \quad \hat{\delta}_c(\vec{k}, t) = \hat{\delta}(\vec{k}, t) \hat{W}_R(k)$$

convolution of δ in real space convolution in Fourier space

• σ_8 measures the "roughness" of the density field

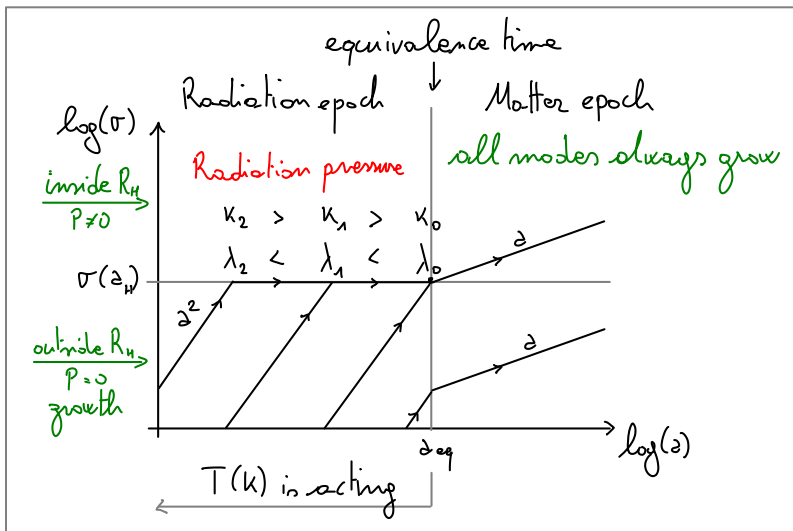
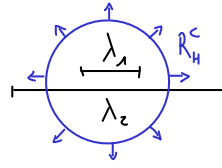
▫ Cosmological parameter σ_8

$W_R = \text{Top hat function, } \boxed{R = 8 \text{ Mpc}/h} \Rightarrow \boxed{\sigma_8}$

CDM power spectrum

• The horizon affects the high k -modes

- During radiation and matter domination, no accelerated expansion $\Rightarrow R_H^c$ grows
- Modes with $\neq \lambda$ enter the horizon at $\neq$ times
- shorter modes $\hat{\delta}_k$ (i.e. larger k) enter R_H earlier
- when $\lambda > R_H^c$ any time = growth
- when $\lambda < R_H^c$ before equivalence time $\Rightarrow$ growth stops because of radiation pressure $P \neq 0$
- " " after " always grow (matter domination, DM, $P=0$)



	$z < z_{eq}$	$z > z_{eq}$
$\lambda < R_H$	$\hat{\delta}_k \propto \text{const}$	$\hat{\delta}_k \propto z$
$\lambda > R_H$	$\hat{\delta}_k \propto z^2$	$\hat{\delta}_k \propto z$
$R_H = \frac{c}{H}$	$R_H \propto z^{3/2}$	$R_H \propto z^2$

• λ_0 threshold for continuous growth if $\lambda > \lambda_0$

$$\lambda_0 = R_H(z_{eq}) \equiv \frac{c}{H} = \frac{c}{H_0 (\Omega_{R_0} \bar{z}_{eq}^{-4} + \Omega_{M_0} \bar{z}_{eq}^{-3})^{1/2}} = \frac{c}{H_0 \sqrt{2\Omega_{M_0}}} \bar{z}_{eq}^{3/2}$$

$$\Omega_{R_0} \bar{z}_{eq}^{-4} = \Omega_{M_0} \bar{z}_{eq}^{-3}$$

$$k_0 \equiv \frac{2\pi}{\lambda} = 2\pi \frac{H_0}{c} \sqrt{\frac{2\Omega_{M_0}}{\bar{z}_{eq}}} = 2\pi \frac{H_0}{c} \Omega_{M_0} \sqrt{\frac{2}{\Omega_{R_0}}}$$

k_0 is known from observations: $H_0, \Omega_{M_0}, \Omega_{R_0}$
(!!)

• Each scale is affected differently according to k

this is quantified by a filter: the transfer function

• Transfer function $T(k)$

- $T(k)$ quantifies the modification of the initial power spectrum due to entrance in R_H
- Initial power spectrum $P_i(k) \propto k^{n_s}$ (from inflation) $n_s = 0.96 \sim 1$
- Today " " $P(k) = P_i(k) T^2(k)$ $T = \text{transfer function (filter)}$

• When does a k mode enter the horizon?

$$\lambda \cdot z_{\text{enter}} = \frac{2\pi}{k} z_{\text{enter}} = R_H \propto \begin{cases} z_{\text{enter}}^2 & (\text{for } z_{\text{enter}} < z_{\text{eq}}) \\ z_{\text{enter}}^{3/2} & (\text{for } z_{\text{enter}} > z_{\text{eq}}) \end{cases}$$

$$\Rightarrow z_{\text{enter}} \propto \begin{cases} k^{-1} & \text{radiation domination} \\ k^{-2} & \text{small } (k \text{ large}) \text{ earlier} \\ & \text{matter domination} \end{cases}$$

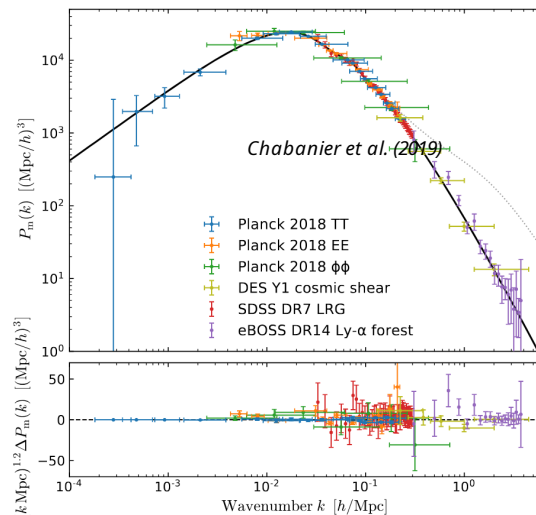
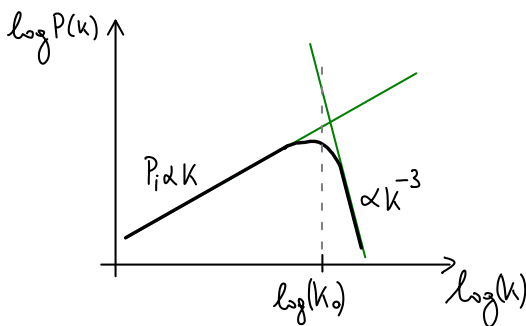
• Relative suppression of growth

for $z < z_{\text{eq}}$ and $\lambda < \lambda_0$ $T(k) = \left(\frac{z_{\text{enter}}}{z_{\text{eq}}} \right)^2 = \left(\frac{k_0}{k} \right)^2$ $\hat{\sigma}_k = \hat{\sigma}_{k_0} \frac{z_{\text{enter}}}{z_{\text{eq}}} = \hat{\sigma}_{k_0} \cdot k^{-2} \Rightarrow \boxed{P(k) = P_i(k) k^{-4}}$

grows up to $\uparrow k$

$$P(k) \propto \begin{cases} k & k < k_0 \\ k^{-3} & k > k_0 \end{cases}$$

CDM power spectrum



BBKS Transfer function (Baendeen-Bond-Kaiser-Szalay, 1986)

Analytic linear perturbation theory of early times

$$T(q) = \frac{\ln(1+2.349)}{2.349} [1 + 3.89q + (16.19)^2 + (5.469)^3 + (6.719)^4]^{-1/4} \quad q = k/T \quad T = \frac{\Omega_m h^2}{(T_{\text{cmb}}/2.7)^2}$$

horizon entry during radiation era
small-scale suppression
radiation/matter domination entry

Assumptions: CDM, No baryons (BAO), No massive neutrinos, NO Dark-energy effects at late times

More modern: Eisenstein & Hu (1998) or CAMB/CLASS Boltzmann solvers

• Cosmological dependencies

$$P_i(k, t) = P_0 k^{n_s} T^2(k) D_+^2(t)$$

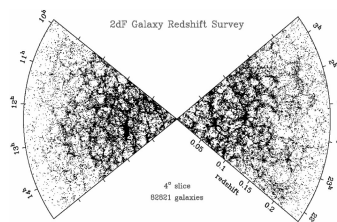
$$\sigma_8 = \left(\frac{d^3 k}{(2\pi)^3} \right)^{1/3} P_i(k, z=0) W_8^2(k)$$

Ω_m affects shape of $P_i(k)$ through $T(k)$ (recall λ_0) and $D_+(t)$

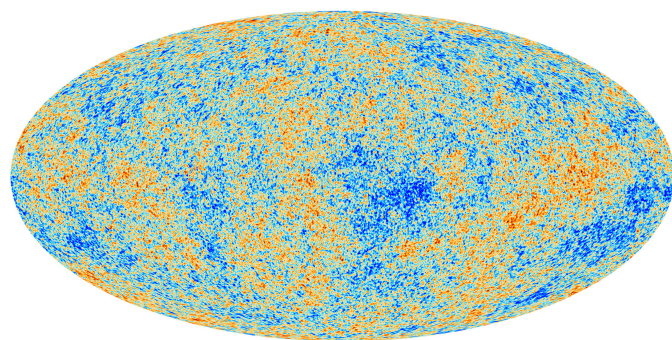
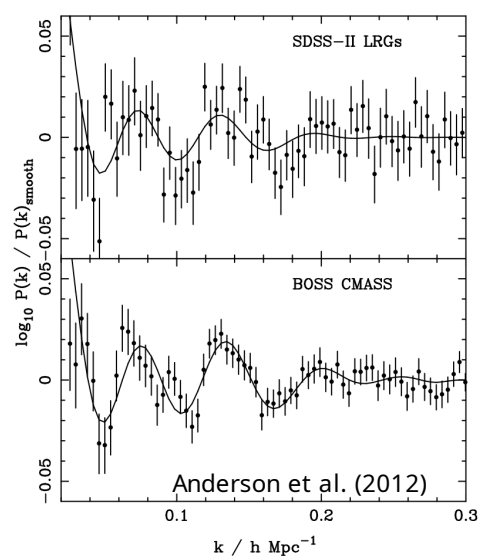
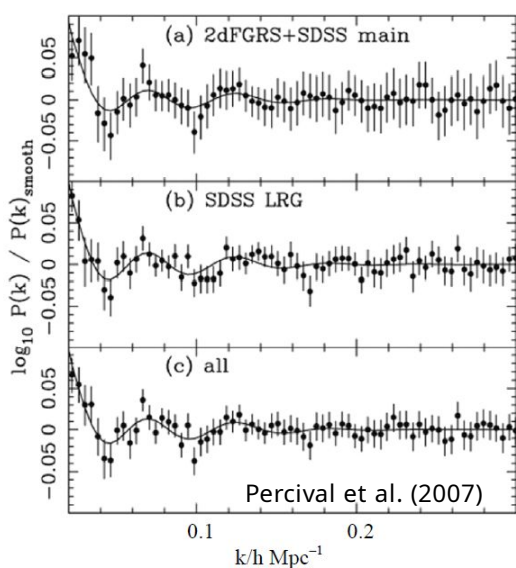
n_s from inflation (steepness of low modes)

Actual spectrum Need to account for :

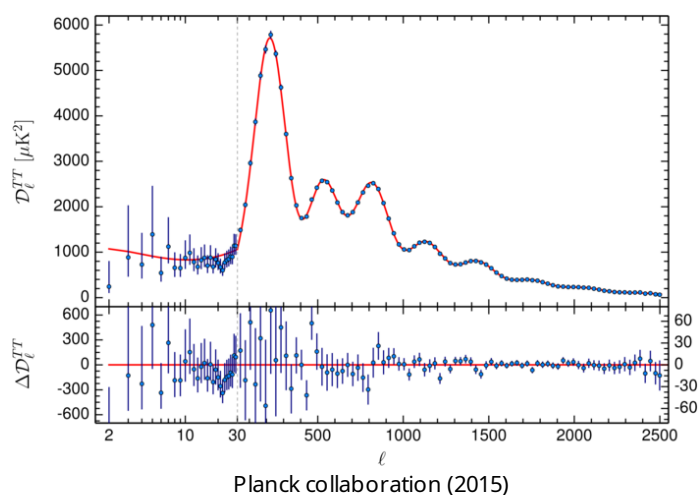
(1) BAOs



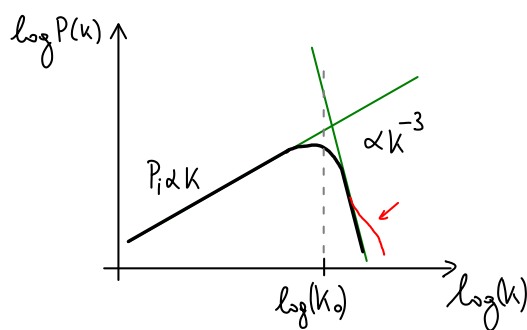
Spectroscopic
galaxy surveys



CMB temperature
fluctuations



(2) Non-linear growth
at large k



Correlation function from galaxy surveys

• Observationally: Use galaxies as density tracers $\xi(\bar{x}, t) \rightarrow n(\bar{x}, t)$

• Consider 2 volumes separated by r : $n_1 dV_1$  r  $n_2 dV_2$

• If random distribution (uncorrelated field):

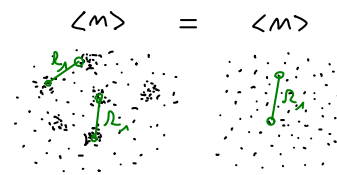


- given a galaxy, the probability to find another galaxy separated by r is the same $\forall r$
- probability to find one galaxy in dV_1 and one in dV_2 is: $dP = n dV_1 \cdot n dV_2 = n^2 dV_1 dV_2$

• For non random distributions: (correlated field)

there is an excess of probability on certain scales

Pairs separated by r are more likely than random



$$\begin{aligned}
 dP &= \langle n_1 dV_1 n_2 dV_2 \rangle = \langle n(1+\delta_1) dV_1 n(1+\delta_2) dV_2 \rangle & n(\bar{x}_i) &= n(1+\delta(\bar{x}_i)) \\
 &= n^2 \langle (1+\delta_1)(1+\delta_2) \rangle dV_1 dV_2 \\
 &= n^2 (1 + \langle \delta_2 \rangle + \langle \delta_1 \rangle + \langle \delta_1 \delta_2 \rangle) dV_1 dV_2 \\
 &= n^2 [1 + \xi(r)] dV_1 dV_2
 \end{aligned}$$

$\Rightarrow$ Correlation function = excess of probability for scale r with respect to random !!

• Volume limited survey

- Count number of pairs of galaxies to estimate $\xi(r)$ in data: $\langle DD \rangle$
- compare it with expected number for a randomized sample: $\langle RR \rangle$

$$\xi = \frac{\langle DD \rangle}{\langle RR \rangle} - 1$$

The "random" is generated with brute force to account for masked areas

(you can not observe where there are bright stars)

The bias factor

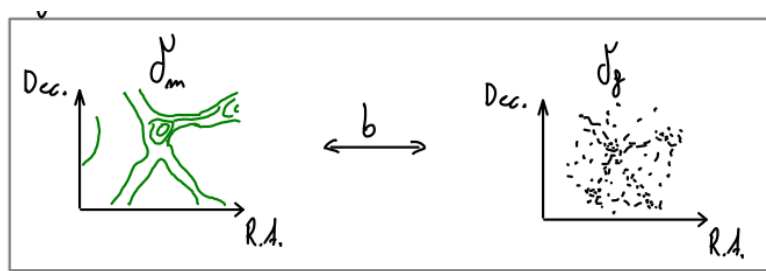
• Caveat: δ (real density contrast) but we use δ_g (visible galaxies!)... they might be different!

$\hookrightarrow$ assume: $\frac{\delta n(\bar{x})}{n} \equiv \delta_g(\bar{x}) = b \cdot \delta(\bar{x}) = \delta(\bar{x}) + (b-1)\delta(\bar{x})$ $b = \text{bias factor}$ ($\approx$ rescaling)

b can be measured via velocity measurements

• bias might be z dependent on z and even the color of galaxies...
red galaxies are more clustered than the blue ones

• You can do the same with the clustering of galaxy clusters



Zel'dovich approximation

- The linearized hydro. equations hold up for $\delta \sim 1$
- For the mildly non-linear regime we can use the Zel'dovich approximation
- Idea: follow the trajectory of individual DM particles longer, in the non-linear regime assume straight trajectories given by the initial gravitational potential, scaled by a time dependent growth. This is a Lagrangian approach (follow particles), not an Eulerian one (fixed space)

$\bar{x}(t) = \bar{q} + \bar{\Psi}(t, \bar{q})$

$\sim \delta \propto a$ $\bar{q}$: Lagrangian initial coordinates ($\propto \bar{q}$ = Hubble flow)

$\bar{\Psi}(\bar{q}, t) = b(t) \bar{\Psi}_0(\bar{q})$: Lagrangian displacement field (peculiar velocities)

$\bar{\Psi}_0(\bar{q}) = \bar{\nabla} \Psi_0(\bar{q}) \propto \bar{\nabla} \Phi_0(\bar{q})$: motion along gradient Ψ_0 = scalar potential
 irrotational $\nabla \times \vec{v} = 0$ (curl free)

$\bar{\nabla}^2 \Psi_0 \propto \bar{\nabla}^2 \Phi_0 \propto D_+ \delta$ shape is fixed, determined by initial Φ_0 / δ
 $b(t) = D_+(t)$ amplitude time evolution (of gravit. potential)

$\dot{\bar{x}} = \dot{D}(t) \bar{\Psi}_0(\bar{q})$ direction of velocity is fixed ($\bar{\Psi}_0$) (no interaction between particles)

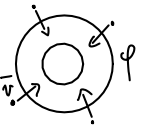
$\bar{v}_0 = \dot{D}(t_0) \bar{\Psi}_0(\bar{q})$ each particle has its own initial velocity

$\frac{d^2 \bar{x}}{dt^2} = 0$ straight (ballistic) trajectory $\bar{x} = \bar{q} + D_+(t) \bar{\Psi}_0(\bar{q})$ $D_+ \sim$ "time variable" $\bar{\Psi}_0 \sim$ velocity

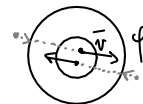
acceleration following a time evolving potential with fixed shape

$\neq \frac{d^2 \bar{q}}{dt^2} \neq 0!$ $\rightarrow$ No orbit bending, No virialization, No multi-stream interaction

$\rightarrow$ It breaks at shell crossing $\det(\delta_{ij} + \delta_{ij} \Psi_i) = 0$ $\delta \sim 1$

- when flow converges $\Rightarrow \delta \uparrow$  $\Rightarrow$ predicts voids separated by "walls": "the pancakes"
- " " diverges $\Rightarrow \delta \downarrow$  Anisotropic structures

- Procedure breaks down when flow lines start crossing each other because their mutual interaction is neglected when they cross

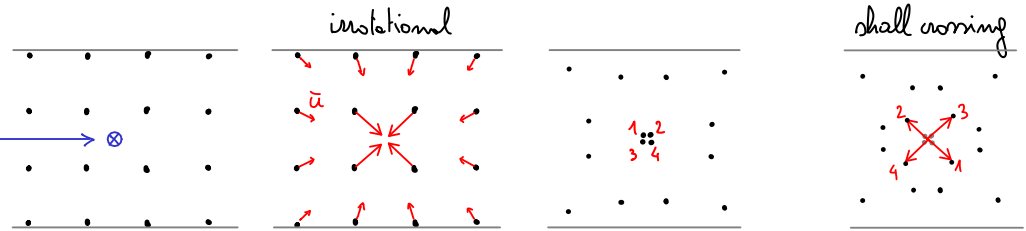


- Then you need a full non-linear treatment

Anisotropic collapse

Matter conservation

Center of a potential well $\rightarrow \otimes$



$\rho(\bar{q}, t) d^3\bar{q} = \rho(\bar{x}, t) d^3\bar{x}$ initial density is "distorted" by particles flow : Jacobian $J_{ij} = \delta_{ij} + \frac{\delta\psi_i}{\delta q_j}$

$$\bar{x}(t) = \bar{q} + b(t) \bar{\Psi}_0(\bar{q})$$

$$\bar{q} \rightarrow \bar{x} \quad \rho = \rho_0 \det^{-1} \left[\frac{\delta x_i}{\delta q_j} \right] = \rho_0 \det^{-1} \left[\delta_{ij} + b(t) \frac{\delta \psi_{0i}(\bar{q})}{\delta q_j} \right] = \rho_0 \left[(1+b\lambda_1)(1+b\lambda_2)(1+b\lambda_3) \right]^{-1}$$

ρ_0 mean density at present time
 $\rho = \rho_0 \bar{a}^3$ matter domination
 δ_{ij} isotropic stretching (diagonal matrix)
 $A_{ij} = \frac{\delta \psi_{0i}}{\delta q_j \delta q_j}$ deformation tensor (changes the "shape")
 $A_{ij} = \frac{\delta \psi_{0i}}{\delta q_j \delta q_j}$ symmetric, hessian of initial potential $\Rightarrow$ real eigenvalues, orthogonal eigenvectors

Density contrast

$$\delta \equiv \frac{\rho - \rho_0}{\rho_0} = \frac{\rho_0 \left[(1+b\lambda_1)(1+b\lambda_2)(1+b\lambda_3) \right]^{-1} - \rho_0}{\rho_0} = \frac{1 - (1+b\lambda_1)(1+b\lambda_2)(1+b\lambda_3)}{(1+b\lambda_1)(1+b\lambda_2)(1+b\lambda_3)}$$

$$\approx 1 - (1+b\lambda_1 + b\lambda_2 + b\lambda_3) = -b(t)(\lambda_1 + \lambda_2 + \lambda_3) \stackrel{\downarrow}{=} -b(t) \bar{\nabla}_q \bar{\Psi}_0(\bar{q}) \quad * \lambda_i = \frac{\delta \psi_{0i}}{\delta q_i}$$

$$|\delta_{ij}| \ll |b f_{ij}|$$

keep 1st order

Proper velocities

$$\dot{\delta} + \bar{\nabla} \bar{u} = 0 \quad \dot{\delta} = -\dot{b} \bar{\nabla}_q \bar{\Psi}_0(\bar{q}) = -\bar{\nabla} \bar{u} \Rightarrow \bar{u} = \dot{b} \bar{\Psi}_0(\bar{q}) = \frac{db}{da} \dot{a} H \bar{\Psi}_0(\bar{q}) = b \frac{d \ln b}{d \ln a} H \bar{\Psi}_0(\bar{q})$$

Time evolution and initial conditions

$$\delta(\bar{z}) = \delta_0 D_+(\bar{z}) \quad \dot{\delta} = \delta_0 \frac{dD_+}{d\bar{z}} \dot{\bar{z}} \stackrel{!}{=} -\dot{b} \bar{\nabla}_q \bar{\Psi}_0(\bar{q}) = \frac{db}{d\bar{z}} \dot{\bar{z}} \bar{\nabla}_q \bar{\Psi}_0(\bar{q}) \Rightarrow \underline{b = D_+} \quad \text{and} \quad \underline{\delta_0(\bar{q}) = -\bar{\nabla} \bar{\Psi}_0(\bar{q})}$$

$$\Rightarrow \begin{cases} \delta_0(\bar{q}) = -\bar{\nabla} \bar{\Psi}_0(\bar{q}) & \bar{\Psi}_0 \propto \bar{\nabla} \phi_0 \\ \bar{u} = D_+ \frac{d \ln D_+}{d \ln a} H \bar{\Psi}_0(\bar{q}) = H D_+(\bar{z}) f(\Omega_m) \bar{\Psi}_0(\bar{q}) & \text{init conditions} \\ \bar{x}(t) = \bar{q} + D_+(t) \bar{\Psi}_0(\bar{q}) = \bar{q} + \frac{\bar{u}}{H f(\Omega_m)} & \text{velocity field} \\ 1 + D_+ \lambda_i = 0 \Rightarrow \rho \rightarrow \infty & \text{straight trajectory} \end{cases}$$

$$\bar{\nabla}^2 \delta \phi_0 = 4\pi G \bar{a}^2 \rho_0 \delta_0 = -4\pi G \bar{a}^2 \rho_0 \bar{\nabla} \bar{\Psi}_0$$

$$\bar{u} \propto \bar{\Psi} \quad f(\Omega_m) = \frac{\delta \ln D_+}{\delta \ln a} \approx \Omega_m^{0.6} \quad \text{see (11)}$$



For that you need a full non-linear treatment

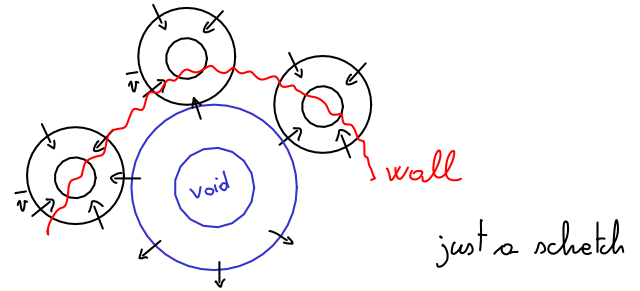
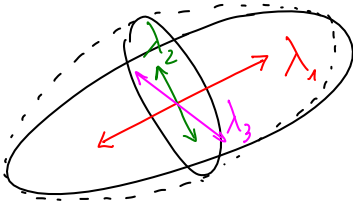
Assume δ_0 to be a Gaussian random field

Probability distribution for the eigenvalues λ_i , multivariate Gaussian:

$$p(\lambda_1, \lambda_2, \lambda_3) = \frac{15^3}{8\pi\sqrt{5}\sigma^6} |(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)| \exp\left\{-\frac{3}{2\sigma^2}[2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) - (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)]\right\}$$

$\Rightarrow P \neq 0$ only if $\lambda_1 \neq \lambda_2$, $\lambda_1 \neq \lambda_3$, $\lambda_2 \neq \lambda_3 \Rightarrow$ the collapse MUST be anisotropic !!

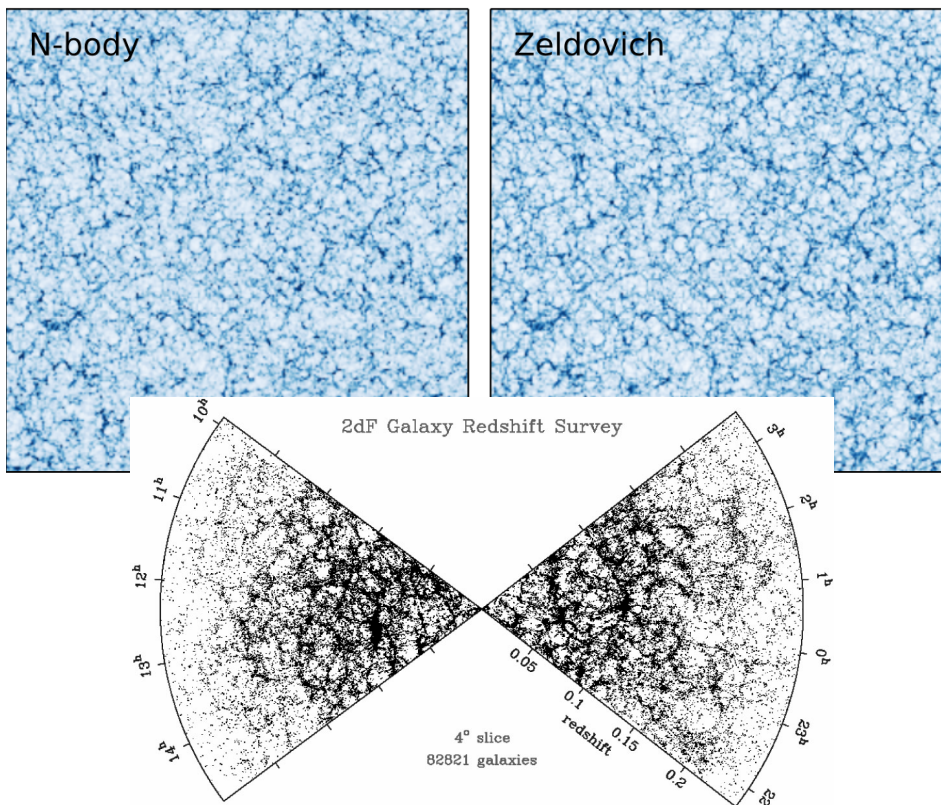
the forming structure flattens as the collapse proceeds : "pancakes"



Sequence of anisotropic collapse (Probability)

- 1) Pancake (sheet) : collapse along one axis λ_1 ($\lambda_1 > \lambda_2 > \lambda_3$)
- 2) Filament : " " two axes
- 3) "Spherical" : " " 3 axes

$\Rightarrow$ Cosmic web dominated by sheets and filaments



M. White

Correlation function in Zel'dovich approximation

Lagrangian perturbation theory (LPT) $\bar{\Psi}(\bar{x}, t) = \bar{\Psi}^{(0)}(\bar{x}, t) + \bar{\Psi}^{(2)}(\bar{x}, t) + \dots$ below $\bar{\Psi}^{(1)} \rightarrow \bar{\Psi}$

Redshift space distortion ---

Biased tracer $\delta_g(\bar{x}, t) = \langle \delta_g \rangle F(\delta_r(\bar{x}))$ $\langle \delta_g \rangle$ mean comoving number density of tracer g (e.g. galaxies)
 F Lagrangian bias function

Correlation function ($F=1$ for simplicity)

$$1 + \xi(\bar{x}) = \int d^3x F(\delta_r(\bar{q})) \int d^3q \bar{\Psi}(\bar{x} - \bar{q}, t) \bar{\Psi}(\bar{q}, t) = \int d^3q \int \frac{d^3k}{(2\pi)^3} e^{i\bar{k}(\bar{x} - \bar{q} - \bar{\Psi}(\bar{q}, t))} \bar{\Psi}(\bar{q}, t)$$

$$1 + \xi(\bar{x} = \bar{x}_2 - \bar{x}_1) = 1 + \langle \delta(\bar{x}_1) \delta(\bar{x}_2) \rangle = \int d^3q_1 \int d^3q_2 \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3} \langle e^{i\bar{k}_1(\bar{x}_1 - \bar{q}_1 - \bar{\Psi}(\bar{q}_1))} e^{i\bar{k}_2(\bar{x}_2 - \bar{q}_2 - \bar{\Psi}(\bar{q}_2))} \rangle$$

$$= \int d^3q_1 \int d^3q_2 \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3} e^{i\bar{k}_1(\bar{x}_1 - \bar{q}_1)} e^{i\bar{k}_2(\bar{x}_2 - \bar{q}_2)} \langle e^{-i\bar{k}_1 \bar{\Psi}(\bar{q}_1)} e^{-i\bar{k}_2 \bar{\Psi}(\bar{q}_2)} \rangle$$

$$= \int d^3q_1 \int d^3q_2 \int \frac{d^3k}{(2\pi)^3} e^{i\bar{k}(\bar{x}_1 - \bar{x}_2 + \bar{q}_1 - \bar{q}_2)} \langle e^{-i\bar{k}(\bar{\Psi}(\bar{q}_1) - \bar{\Psi}(\bar{q}_2))} \rangle$$

Isotropy $\delta_0(k_1 + k_2)$, $\bar{q} = \bar{q}_1 - \bar{q}_2$, $\bar{Q} = \frac{1}{2}(\bar{q}_1 + \bar{q}_2) \rightarrow$

$$= \int d^3q \int \frac{d^3k}{(2\pi)^3} e^{i\bar{k}(\bar{q} - \bar{r})} \langle e^{-i\bar{k} \Delta \bar{\Psi}} \rangle \quad \Delta \bar{\Psi} \equiv \bar{\Psi}(\bar{q}_1) - \bar{\Psi}(\bar{q}_2)$$

Cumulant theorem $\rightarrow$

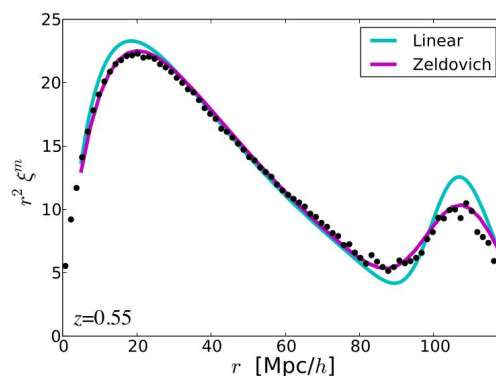
$$= \int d^3q \int \frac{d^3k}{(2\pi)^3} e^{i\bar{k}(\bar{q} - \bar{r})} e^{-\frac{i}{2} \bar{k}^i \bar{k}^j \frac{\langle \Delta \bar{\Psi}_i \Delta \bar{\Psi}_j \rangle}{A_{ij}(\bar{q})}} \quad A_{ij}(\bar{q}) = \langle \Delta \bar{\Psi}_i \Delta \bar{\Psi}_j \rangle$$

Analytical Gaussian integral $d\mathbf{k} \rightarrow$

$$= \int d^3q \frac{1}{(2\pi)^{\frac{3}{2}} \sqrt{\det(A)}} \exp\left[-\frac{1}{2}(\bar{r} - \bar{q})^T A^{-1}(\bar{r} - \bar{q})\right] \quad A_{ij}(\bar{q}, t) = D_+(t) A_{ij}^{(0)}(\bar{q})$$

Note: $\Delta \bar{\Psi} \equiv \bar{\Psi}_1 - \bar{\Psi}_2$: translational invariance (homogeneity)

$\bar{r} \equiv \bar{q}_1 - \bar{q}_2$: relative displacements, not absolute ones



• isotropy: $\langle \delta(\mathbf{k}_1) \delta(\mathbf{k}_2) \rangle \propto \delta_0(\mathbf{k}_1 + \mathbf{k}_2) \Rightarrow \mathbf{k}_2 = -\mathbf{k}_1$

• $\bar{\Psi}$ is Gaussian $\Rightarrow$ Cumulant theorem: $\langle \bar{\Psi} \rangle = 0$, $\langle \exp(\bar{\Psi}) \rangle = \exp(-\frac{1}{2} \langle \bar{\Psi}^2 \rangle) = e^{-\frac{i}{2}(\bar{\mathbf{k}}^T A \bar{\mathbf{k}})}$

$$A_{ij} \equiv \langle (\bar{\Psi}_i(\bar{q}_1) - \bar{\Psi}_i(\bar{q}_2)) (\bar{\Psi}_j(\bar{q}_2) - \bar{\Psi}_j(\bar{q}_1)) \rangle = \sigma_{\pm}^2(q) \delta_{ij} + [\sigma_r^2(q) - \sigma_{\pm}^2(q)] \hat{q}_1 \hat{q}_2 \quad \sigma_{\pm} = \int \frac{dk}{k} \frac{\Delta^2(k)}{k^2} \left[\frac{2}{3} - \frac{2\beta_1(kq)}{kq} \right]$$

• Solve integral numerically

Spherical collapse and dark matter haloes

- Matter dominated spherical overdensity : It behaves as a universe on its own
- A closed "universe" collapses : matter in universe "organized" in Dark-Matter Haloes

Condition for the collapse of a perturbation

$$\rho_p(t_i) = \rho(t_i)(1 + \delta_i) \quad \text{collapse} \quad \text{linear criteria}$$

$$\rho_p(t_i) = \rho(t_i)(1 + \delta_i) > 1 \Rightarrow \delta_i > \frac{1 - \Omega(t_i)}{\Omega(t_i)}$$

t_i = time of collapse perturbation universe

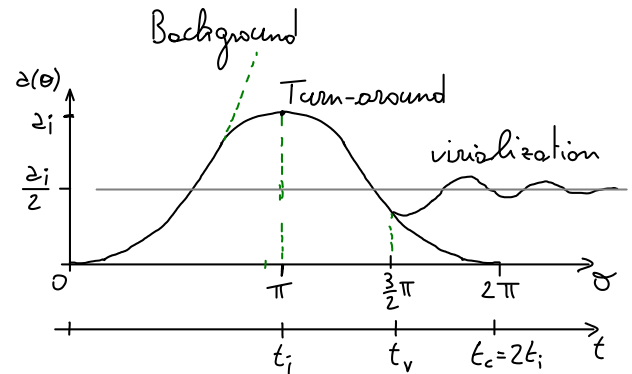
if $\begin{cases} 1 - \Omega(t_i) < 0 & (\Omega > 1) \\ 1 - \Omega(t_i) > 0 & (\Omega < 1) \end{cases}$ but $\delta_i \geq 0$ by construction $\Rightarrow$ collapse is always possible
 $\Rightarrow$ there is a minimum δ_i for collapse to happen

Time and radius of region of collapse

Solve Friedmann eq. for matter domination

$$a(\theta) = \frac{a_0}{2} \frac{\Omega_{p0}}{(\Omega_{p0} - 1)} (1 + \cos \theta)$$

$$t(\theta) = \frac{1}{2H_0} \frac{\Omega_{p0}}{(\Omega_{p0} - 1)^{3/2}} (\theta - \sin \theta)$$



Collapse starts at $a = \max$, i.e. $\theta = \pi$

$$a_i = a_0 \frac{\Omega_{p0}}{(\Omega_{p0} - 1)}$$

$$t_i = \frac{\pi}{2H_0} \frac{\Omega_{p0}}{(\Omega_{p0} - 1)^{3/2}}$$

$$(1) \quad \left(\frac{a_i}{a_0}\right) = \frac{\Omega_{p0}}{(\Omega_{p0} - 1)}$$

$$(2) \quad \Omega_p = \frac{2H_0}{\pi} (\Omega_{p0} - 1)^{3/2} t_i$$

Overdensity at collapse

$$\rho_p(t_i) = \rho_{p0} \left(\frac{a_i}{a_0}\right)^{-3} = \rho_{p0} \rho_{m0} \cdot \frac{\Omega_{p0}^{-2}}{(\Omega_{p0} - 1)^{-3}} = \rho_{m0} \frac{\pi^2 (\Omega_{p0} - 1)^3}{4H_0^2 (\Omega_{p0} - 1)^3 t_i^2} = \frac{3H_0^2}{8\pi G} \frac{\pi^2}{4H_0^2 t_i^2} = \frac{3\pi}{32G t_i^2}$$

Density at virialization

$$E_i = E_{pot,i} = \frac{GM^2}{R_i}$$

$$E_v = E_{pot,v} + E_{kin,v} = E_{pot,v} - \frac{1}{2} E_{pot,v} = \frac{1}{2} \frac{GM^2}{R_v}$$

virial theorem $E_{pot,v} + 2E_{kin,v} = 0$

at collapse virialization
 $\Rightarrow E_i = E_v \quad R_v = \frac{1}{2} R_i$

$$\rho_p(t_v) = 8 \rho_p(t_i)$$

$\uparrow$
 R^3

• Evolution of the overdensity

▫ Background

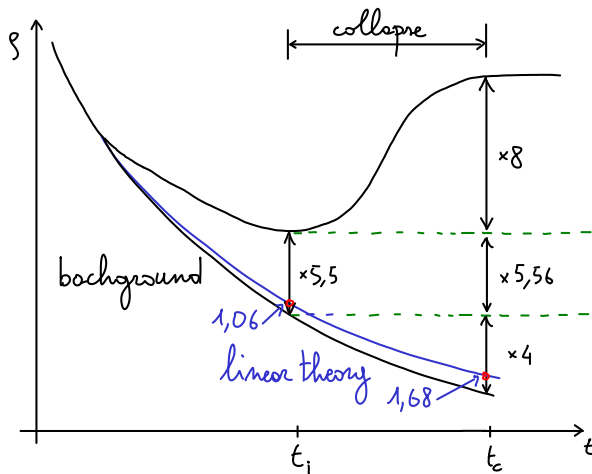
at turn-around t_i $\rho_{Bi} = \rho_{a,0} \bar{a}_i^{-3}$ $a_i = \left(\frac{3}{2} H_0 t_i \right)^{2/3} = \left(\frac{3}{2} H_0 \frac{\pi}{2 H_0} \frac{\Omega_{p,0}}{(\Omega_{p,0}-1)^{3/2}} \right)^{2/3}$
 (1)

▫ Overdensity $\delta = \frac{\rho_i}{\rho_b} - 1 \equiv \chi - 1$

- at turnaround: t_i $\chi(t_i) = \frac{\rho_{pi}}{\rho_{Bi}} = \frac{\rho_{a,0} (\Omega_{p,0}-1)^3}{\Omega_{p,0}^2} \frac{1}{\rho_{a,0}} \left(\frac{3\pi}{4} \frac{\Omega_{p,0}}{(\Omega_{p,0}-1)^{3/2}} \right)^2 = \frac{9\pi^2}{16} \approx 5,56$

- at collapse: t_c $\chi_c = 8 \chi_i^2 \approx 178$
 from density at virialization $\uparrow$ $\uparrow$ because of further drop in $\rho_b \downarrow$ $t_c = 2t_i$ in eq. (1)

- at virialization: t_v $\chi \approx 145$ "this value is never used"



Linear theory gives very wrong values at t_i, t_c but it can still be used to set "land marks"

• Linear criteria for collapse

- Trick: map the timing of the spherical collapse on the one given by the linear theory
- $\Rightarrow$ at turn around t_i : $\delta_{lin} \approx 1,06$ when structure begins to collapse (wrong by a factor of 4,5)
- $\Rightarrow$ at collapse t_c : $\delta_{lin} \approx 1,69$ " " is collapsed (against the non-linear 178)

$\hookrightarrow$ very small dependency on cosmological parameters

Including $\Omega_{\Lambda,0} = 1 - \Omega_{m,0}$ (flat): $\delta_c = 1,69 \cdot (1 + 0,0123 \log_{10} \Omega_{m,0})$

$\Rightarrow \delta_{c,lin} \approx 1,69$ can use this number for all cosmologies

- Advantage: link to statistic of random fields

$\Rightarrow$ Power spectrum quantifies how many regions reach $\delta_{c,lin}(\bar{x}, t)$

$\Rightarrow$ derive Mass function of dark matter haloes

Press-Schechter mass function

Assumptions:

(1) Direct collapse (no merging between haloes)

(2) Use linear perturbation theory:

↳ Random density field

$$\delta(\vec{x}, t)$$

↳ Linear criteria for collapse

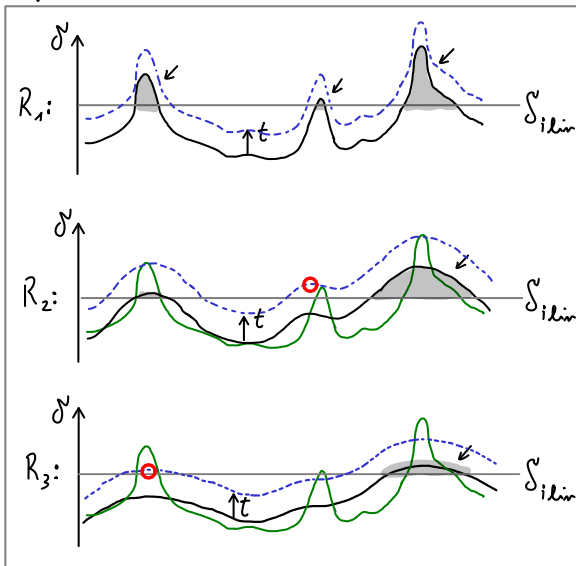
$$\delta(\vec{x}, t) > \delta_{lin}$$

↳ Relate mass to scales

$$M = \frac{4\pi}{3} \rho_B R^3 \quad \sigma_R^2 = 4\pi \int \frac{k^2 dk}{(2\pi)^3} P(k) \hat{W}_R^2(k)$$

Top hat W_R to link k (i.e. scale λ) to M within R

Different scales cross δ_{lin} at different times



- $\delta \rightarrow \delta_R$ convolved

- e.g. 3 scales: $R_1 < R_2 < R_3$

$$k_1 > k_2 > k_3$$

$$\sigma_1 < \sigma_2 < \sigma_3$$

- $t \uparrow$ time evolution: $\delta_R = \delta_{lin} D_+(t)$ (linear growth)

$$\sigma_R(z) = \sigma_R D_+(z)$$

- smaller structures form first

■ cross δ_{lin} first, ● cross at a later time

↳ Hierarchical structure formation (!)

Probability distribution to have a certain δ_R

$$p_R(\delta, z) = \frac{1}{\sqrt{2\pi} \sigma_R(z)} \exp\left(-\frac{\delta^2}{2\sigma_R^2(z)}\right) \quad (\text{Gaussianity})$$

Probability to exceed δ_{lin}

$$F(R, z) = \int_{\delta_{lin}}^{\infty} p_R(\delta, z) d\delta = \frac{1}{2} \text{erfc}\left(\frac{\delta_{lin}}{\sqrt{2} \sigma_R(z)}\right)$$

erfc: complementary error function

Distribution of haloes over mass

$$\text{use: } \frac{d}{dM} = \frac{d}{d\sigma_R} \frac{d\sigma_R}{dM} \quad \text{and} \quad \frac{d}{dx} \text{erfc}(x) = -\frac{2}{\sqrt{\pi}} e^{-x^2}$$

$$\Rightarrow \frac{\delta F}{\delta M} = \frac{1}{2} \cdot \frac{2}{\sqrt{\pi}} \exp\left(-\frac{\delta_{lin}^2}{2\sigma_R^2(z)}\right) \cdot \frac{\delta_{lin}}{\sqrt{2} \sigma_R^2(z)} \cdot \frac{d\sigma_R}{dM} = \frac{\delta_{lin}}{\sqrt{2\pi} \sigma_R(z)} \exp\left(-\frac{\delta_{lin}^2}{2\sigma_R^2(z)}\right) \frac{d \ln \sigma_R}{dM}$$

• Fix normalization :

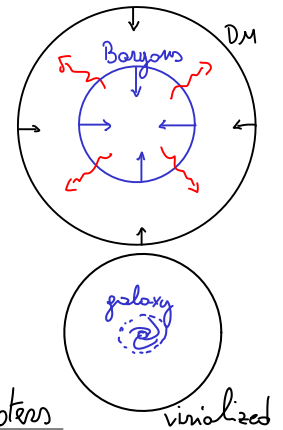
$$\int_0^{\infty} \frac{\delta F(M, z)}{\delta M} dM \stackrel{!}{=} \frac{1}{2} \quad \text{because prescription do not account for decreasing modes that have } \delta > \delta_{lim} \text{ already}$$

• Press-Schechter mass function (comoving number density)

$$\begin{aligned} N(M, z) dM &= \frac{\delta F}{\delta M} \cdot 2 \cdot \frac{1}{V} = \sqrt{\frac{2}{\pi}} \frac{\delta_{lim}}{\sigma_R(z)} \exp\left(\frac{-\delta_{lim}^2}{2\sigma_R^2(z)}\right) \frac{d \ln \sigma_R}{d \ln M} M \cdot \frac{1}{V} \quad M = \bar{\rho}_B V \\ &= \sqrt{\frac{2}{\pi}} \frac{\bar{\rho}_B \delta_{lim}}{\sigma_R(z)} \exp\left(\frac{-\delta_{lim}^2}{2\sigma_R^2(z)}\right) \frac{d \ln \sigma_R}{d \ln M} \quad (!!) \\ &\quad \uparrow \sigma_R D_+(z) \end{aligned}$$

• Galaxies and galaxy clusters

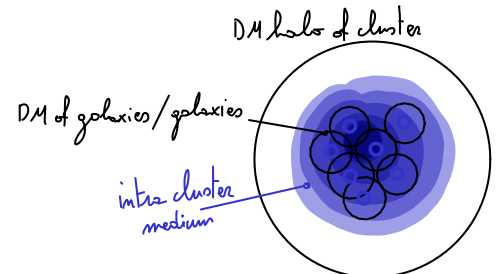
- Baryons will participate to the gravitational collapse
- They will collapse more because they can radiate away energy
- Star will form from the further collapse of local substructures
- Galaxies are formed
- Smaller dark matter haloes (galaxies) will merge to form galaxy clusters



• Constraining the cosmological parameters

1) Clusters counts

- detect dark matter haloes (galaxy clusters) and
 - optical : identify the galaxies they contain
 - X-rays : Bremsstrahlung emission from intra cluster medium (ICM)
 - Millimetric : Sunjaev-Zeldovich effect (compton scattering of CMB photons with ICM free e^-)
 - Gravitational lensing : shear peaches
- count them as a func of M and z
 - $\Rightarrow$ Constraints on Ω_m, σ_8 and geometry (Ω_Λ) through V



2) Clustering

- of galaxy
- of galaxy clusters

Properties of DM haloes

- M_{200}, R_{200} : mass, M_{200} , within a radius, R_{200} , for which $\bar{\rho} = 200 \bar{\rho}_c$

$$M_{200} = 200 \bar{\rho}_c \frac{4\pi}{3} R_{200}^3 = 200 \frac{100 H^2}{8\pi G} \frac{4\pi}{3} R_{200}^3 = \frac{100 H^2 R_{200}^3}{G}$$

$$R_{200} = \left(\frac{G M_{200}}{100 H^2} \right)^{1/3}$$

200 comes from $\kappa_c = 178$ of spherical collapse

- M_{200m} : alternative definition based on the mean density: $M_{200m} = 200 \bar{\rho}_m \frac{4\pi}{3} R_{200}^3$

this definition is closer to a dynamical definition such as M_{vir} but requires R_m

- M_{vir}, R_{vir} : dynamical motivation from virial theorem, $M_{200m} \approx M_{vir}$

(these definitions do not require to specify the radial density profile)

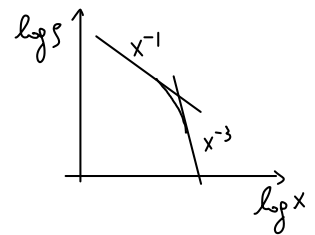
▪ Radial profile of haloes

- Fit to N -body numerical simulations

Navarro-Frenk-White (NFW) profile

$$\rho(r) = \frac{\rho_s}{x(1+x)^2} \quad x \equiv \frac{r}{r_s}$$

well supported by observations: gravitational lensing, radial velocities, ...



Concentration

$$c \equiv \frac{R_{200}}{r_s}$$

$c(M_{200}, z, \bar{\Sigma})$ because of the hierarchical growth:

smaller haloes have higher c because they formed earlier when $\bar{\rho}_c$ was higher

$\Rightarrow$ (the same mass was contained in a smaller radius)

Mass within r :

$$M(r) = 4\pi \int_0^r \rho(r') dr' = 4\pi r_s^3 \int_0^x \rho(x') dx' = 4\pi \rho_s r_s^3 \int_0^x \frac{x' dx'}{(1+x')^2} = 4\pi \rho_s r_s^3 \left[\ln(1+x) - \frac{x}{1+x} \right]$$

set r_s based on $M(R_{200}) \Rightarrow$ Profile fully determined by 1 parameter (mass)

- Careful! $\lim_{r \rightarrow \infty} M(r) = \infty$ logarithmically divergent

$\lim_{r \rightarrow 0} \rho(r) = \infty$ density diverges at zero, never mind... in any case one must account for baryons that in the core are relevant!

- Other profiles: generalized NFW, truncate NFW, Einasto profile

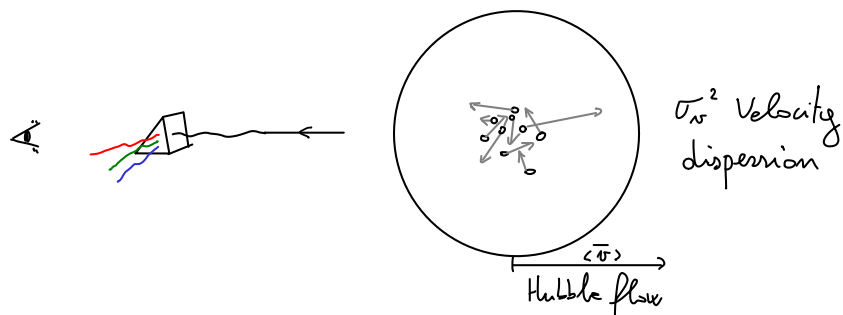
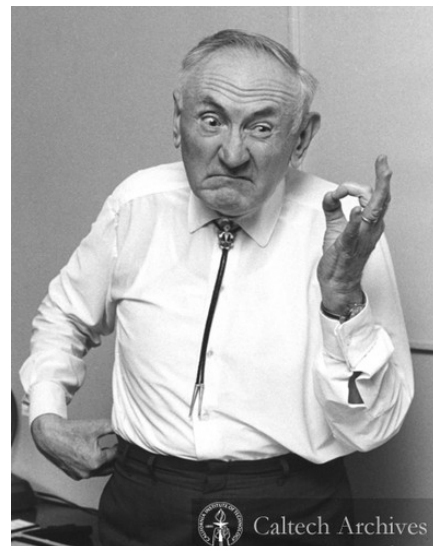
Dark matter in galaxy clusters

The first to speak about "dunkle Materie" Fritz Zwicky (1933)

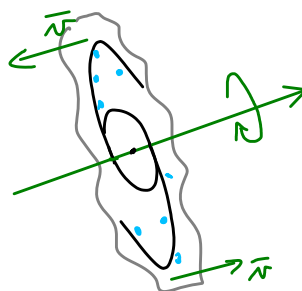
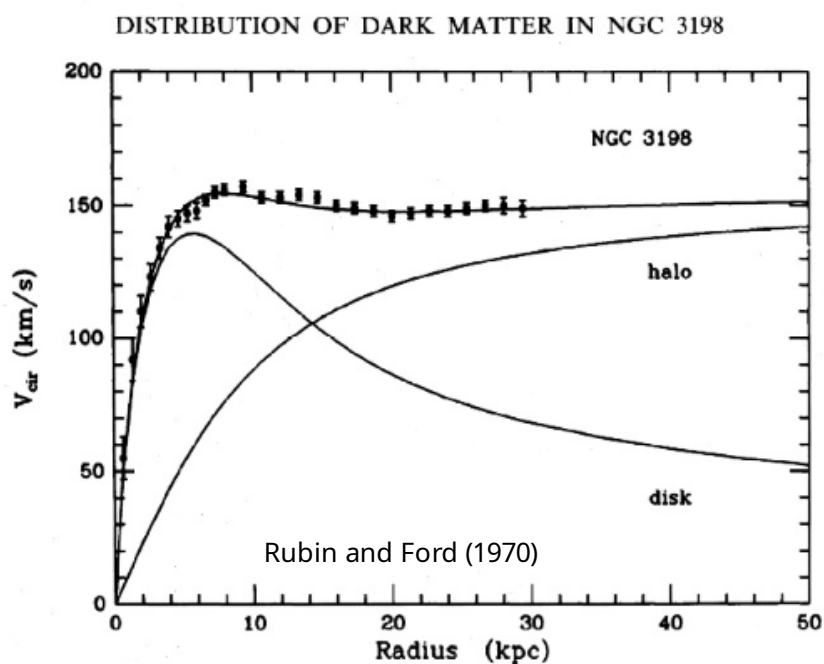
Velocity dispersion measure of Coma cluster

Virial theorem

He measured 400 times more mass than just the visible one



Dark matter in galaxies



Galaxy Clusters

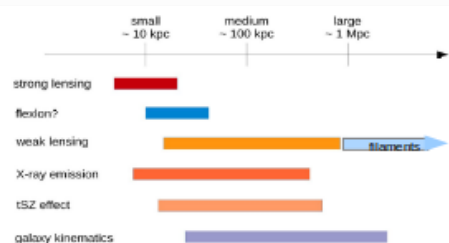
What they are, how to use them

Properties:

Mass $M_v \sim 10^{14} - 10^{15} M_\odot$
 Virial radius $r_v \sim 1 - 3 \text{ Mpc}$

- + Dark-matter halo
- + gas $T \sim 10^7 - 10^8 \text{ K}$
- + stars & galaxies

Different regions
 Different response to distance

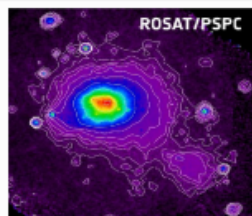


(1) Stars / galaxies

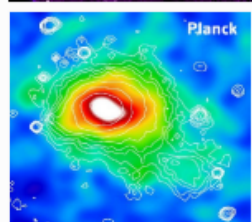


Optical emission from galaxies

(2) ICM Temperature $T \sim 10^7 - 10^8 \text{ K}$



Bremsstrahlung emission
 - X-ray luminosity $L_x \approx 10^{45} \text{ erg/s}$



Inverse Compton scattering of CMB photons
 - tSZ effect (CMB) $\Delta T_{\text{tSZ}} \sim 100 \mu\text{K}$
 - kSZ effect (CMB) $\Delta T_{\text{kSZ}} \sim 10 - 30 \mu\text{K}$

(3) Dark matter + baryons



Gravitational Lensing (background galaxies & CMB)
 - strong $R_E \approx 5''$
 - weak

An example of cosmology based on galaxy clusters: AMICO-KiDS-DR3

Number count of galaxy lclusters: Lesci et al. (2022a)

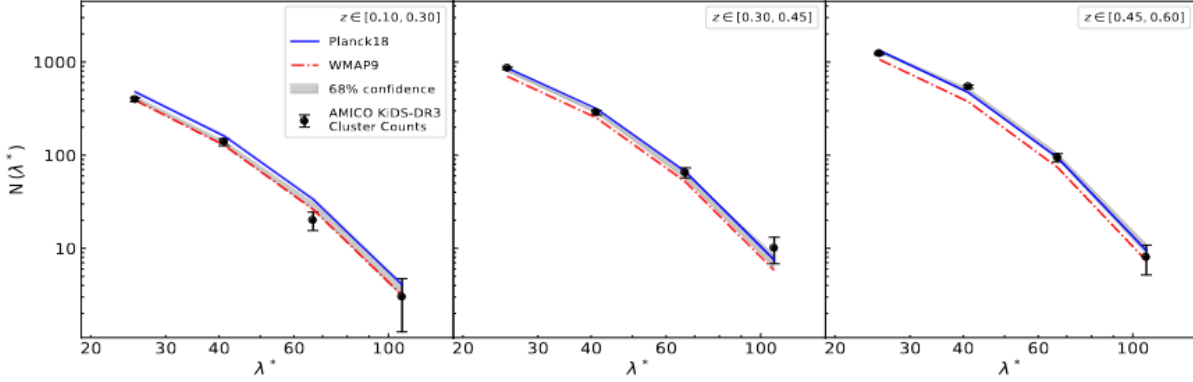


Fig. 4. Number counts from the AMICO KiDS-DR3 cluster catalogue as a function of the intrinsic richness λ^* , in the redshift bins $z \in [0.10, 0.30]$, $z \in [0.30, 0.45]$, and $z \in [0.45, 0.60]$, from left to right. The black dots represent the counts directly retrieved from the catalogue, where the error bars are given by the Poissonian noise. The solid blue lines represent the model computed by assuming the cosmological parameters obtained by Planck Collaboration VI (2020) (Table 2, TT, TE, and EE+lowE), while the red dashed lines show the results based on the WMAP cosmological parameters (Hinshaw et al. 2013) (Table 3, WMAP-only Nine-year). Both in the Planck and WMAP cases, the scaling relation parameters and the intrinsic scatter have been fixed to the median values listed in Table 2, retrieved from the modelling. The grey bands represent the 68% confidence level derived from the multivariate posterior of all the free parameters considered in the cosmological analysis.

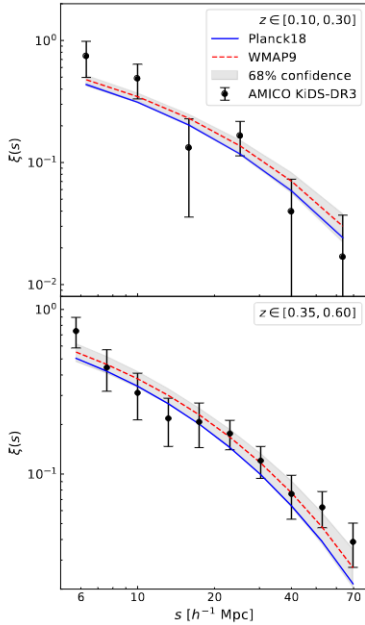


Fig. 2. Redshift-space 2PCF (black dots) of the AMICO KiDS-DR3 clusters in the spatial range $s \in [5, 80] h^{-1} \text{Mpc}$, and redshift ranges $z \in [0.10, 0.30]$ (top panel) and $z \in [0.35, 0.60]$ (bottom panel). In both panels, the grey band represents the 68% confidence level derived from the multivariate posterior of the free parameters considered in the cosmological analysis, described in Sect. 4.1. The model computed by assuming the cosmological parameters derived by Planck (Planck Collaboration VI 2020, Table 2, TT, TE, and EE+lowE; blue lines) and WMAP (Hinshaw et al. 2013, Table 3, WMAP-only Nine-year; red lines) is represented by the solid blue lines and by the dashed red lines, respectively. In both cases, the median values of the scaling relation parameters derived by Lesci et al. (2022) are assumed.

Clustering of galaxy clusters: Lesci et al. (2022b)

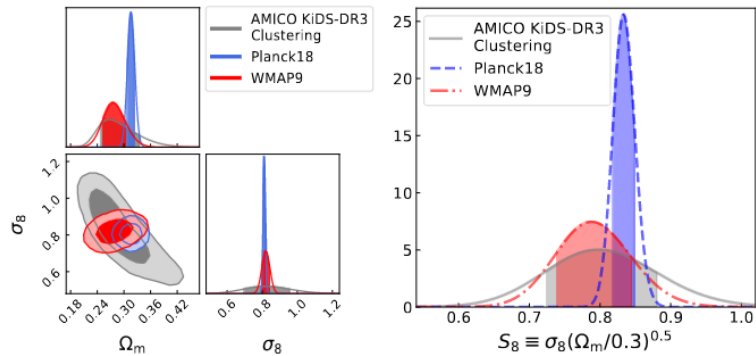
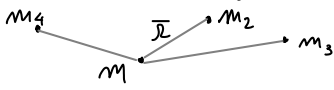


Fig. 3. Constraints obtained from the cosmological analysis, compared to WMAP and Planck results. In the left panel, we show the 68% and 95% confidence levels in the $\Omega_m - \sigma_8$ parameter space, along with the 1D marginalised posteriors with the relative intervals between the 16th and 84th percentiles, in the case of the cluster clustering analysis of the AMICO KiDS-DR3 catalogue (grey lines). In the same panel, we also display the results from WMAP (Hinshaw et al. 2013, Table 3, WMAP-only Nine-year; red lines) and Planck (Planck Collaboration VI 2020, Table 2, TT, TE, and EE+lowE; blue lines). In the right panel, we show the posteriors for the parameter S_8 , where the bands show the intervals between 16th and 84th percentiles. The colours are the same as in the left panel.

N-Body numerical simulations

- Gravitational interaction only $\Rightarrow$ DM (not collisional)
- Matter field is composed by "macro-particles" e.g. $m \approx 10^{10} M_{\text{sun}}/h$

- Solve for :

$$\begin{cases} \dot{\vec{v}} = \vec{F}/m \\ \dot{\vec{x}} = \vec{v} \\ \nabla^2 \phi = 4\pi G \rho \end{cases} \quad \vec{F} = \text{force given by all particles}$$


- Discretize time steps

$$\begin{cases} \vec{v}(t+\Delta t) \stackrel{!}{=} \vec{v}_{\text{new}} = \vec{v}(t) + \frac{\vec{F}}{m} \Delta t & (\text{Eulerian approach}) \\ \vec{x}(t+\Delta t) \stackrel{!}{=} \vec{x}_{\text{new}} = \vec{x}(t) + \vec{v}_{\text{new}} \Delta t \end{cases}$$

$$\Delta t = \sqrt{\epsilon / |\vec{F}/m|} \quad \text{optimal time step}$$

$\uparrow$ acceleration
 $\uparrow$ length (the spatial resolution you care about)

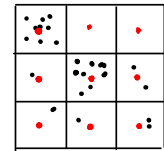
• Different schemes to evaluate $\vec{F}$

A) Particle-Particle model (PP)

$$\vec{F}_i = \sum_{j \neq i}^N \vec{F}_{ij} = \sum_{j \neq i}^N \frac{G m_i m_j}{|\vec{r}_{ij}|^3} \vec{r}_{ij} \quad (N^2) \quad \text{Slow but precise}$$

B) Particle-Mesh (PM)

- Divide the volume in cells and get density ρ
- Resolution is fixed by the grid ($N \log N$)



- Fourier space $\Rightarrow$ FFT

$$\begin{cases} \rho_k(\vec{x}) = \int_{-\infty}^{\infty} \rho(\vec{x}) e^{-i\vec{k} \cdot \vec{x}} \frac{d^3 k}{(2\pi)^3} \\ \nabla^2 \phi = 4\pi G \rho \rightarrow \tilde{\phi}_k = \rho_k / k^2 \end{cases} \quad \text{Newtonian, Green's function } 1/k^2$$

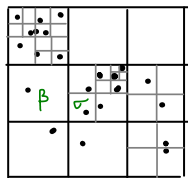
- Back to real space to get $\phi(\vec{x})$
- Apply force to all particles and repeat

c) P^3M (PP-PM)

- Hybrid approach
- Large range force use PM, short range force use PP $\Rightarrow \vec{F} = \overset{PP}{\vec{F}_{short}} + \overset{PM}{\vec{F}_{long}}$
- Same precision of PP but faster

D) Tree-Code (PT)

- Use Gauss theorem
- Collective force of a distant collection of particles $\rightarrow$ Baricenter !
- More precise than PM but a bit slower
- computation time scales as $\sim N \log N$



- 1) Build the tree
-
- split until you have 1 particle per cell

- 2) Compute baricenter and mass of all nodes

- 3) " force of all modes on a particle

$\hookrightarrow$ eg. particle p feels force from 6 cells (the largest ones)
 " σ " " " 9 "

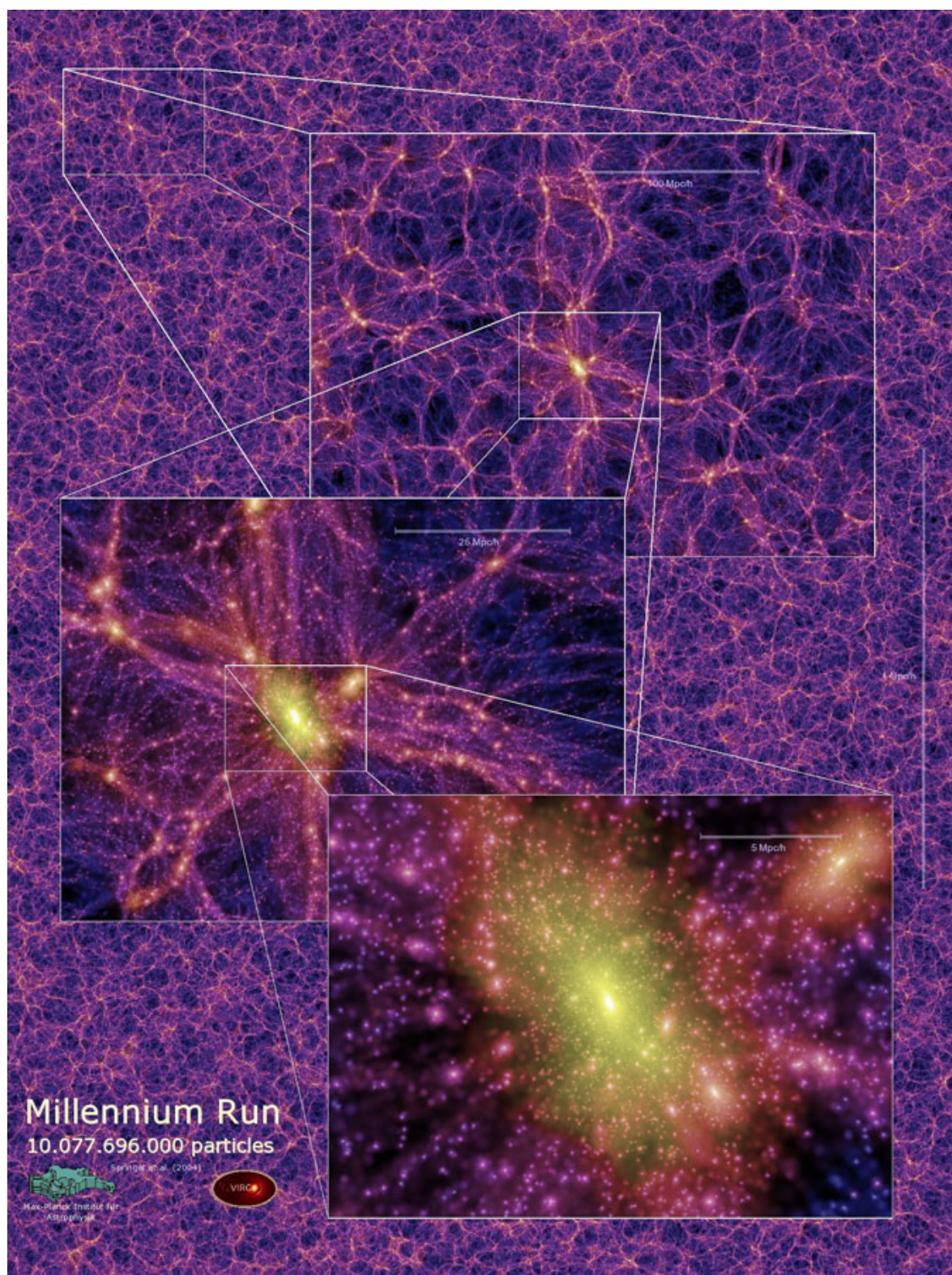
E) Tree PM (gadget)

$$\varphi_{\bar{k}} = \varphi_{\bar{k}}^{\text{short}} + \varphi_{\bar{k}}^{\text{long}}$$

$$\psi_{\bar{u}}^{\text{long}} = \psi_{\bar{u}} \exp(-\bar{u}^2 \bar{R}_5) \quad \text{PM}$$

$$f_k^{\text{short}} = f_k [1 - \exp(-\bar{k}^2 \bar{n}_s)] \quad \text{Tree}$$

Example DM only



Key results

Dark Matter haloes:

- NFW profile / Einasto profile

- Navarro, Frenk + White: haloes in n -body simulation show a profile:

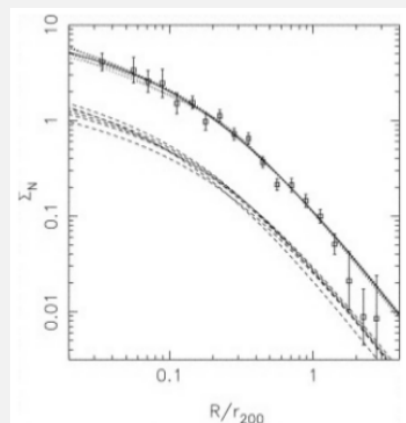
$$\rho \propto \frac{1}{x(1+x^2)} \quad \text{with} \quad x \equiv \frac{r}{r_c} \quad \text{and} \quad r_c = cr_{\text{vir}} \quad (10)$$

- universal density profile, applicable to haloes of all masses
- fitting formula breaks down:
 - infinite core density
 - total mass diverges logarithmically



Einasto profile

Flat rotational curves



The Mass functions

- Press & Schechter (1974)
- Sheth & Tormen (1999)
- Jenkins et al. (2001)
- Tinker et al. (2008)

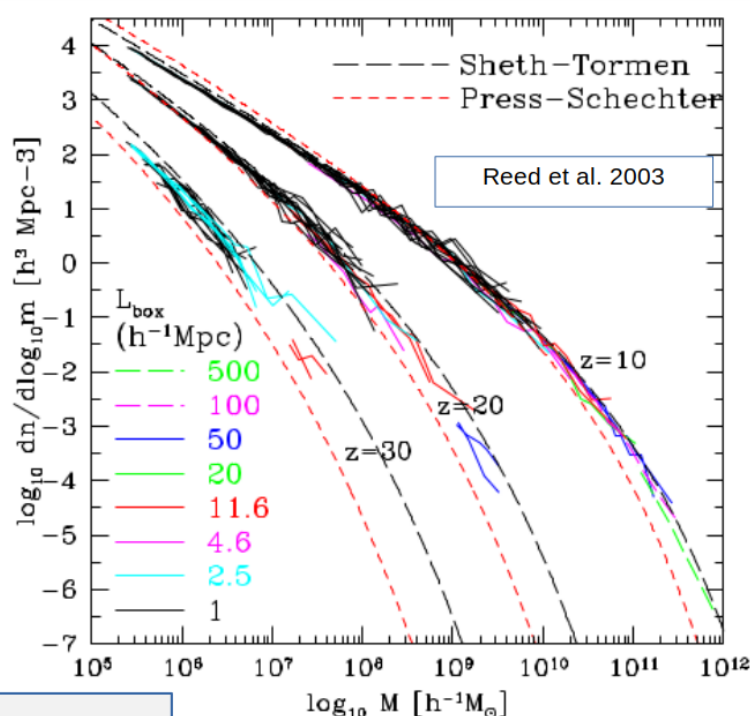
Press & Schechter (1974)

$$f_{\text{P-S}}(\sigma) = \sqrt{\frac{2}{\pi}} \frac{\delta_c}{\sigma} \exp \left[-\frac{\delta_c^2}{2\sigma^2} \right]$$

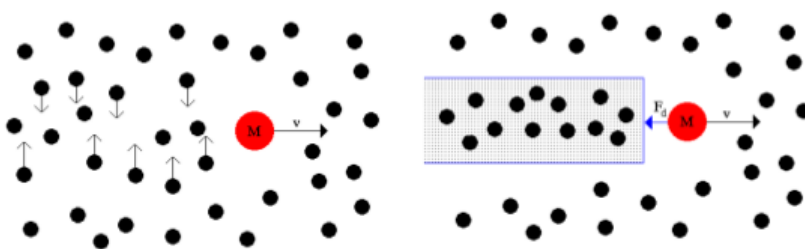
Sheth & Tormen (1999)

$$f_{\text{S-T}}(\sigma) = A \sqrt{\frac{2a}{\pi}} \left[1 + \left(\frac{\sigma^2}{a\delta_c^2} \right)^p \right] \frac{\delta_c}{\sigma} \exp \left[-\frac{a\delta_c^2}{2\sigma^2} \right]. \quad (5)$$

The choice of values $A = 0.3222$, $a = 0.707$ and $p = 0.3$ provides a significantly better fit to mass functions determined



They capture non-linear effects such as dynamical friction



source: J. Schombert

Baryons: hydrodynamical simulations

• Baryons: collisional $\Rightarrow P \neq 0$ need to account for it

$$\begin{cases} \frac{\delta \rho}{\delta t} + \frac{1}{a} \bar{\nabla}(\rho \bar{v}) = 0 & \text{matter conservation} \\ \frac{\delta(\rho \bar{v})}{\delta t} + \frac{1}{a} \bar{\nabla}(\rho \bar{v}^2 + \bar{P}) = -\frac{\dot{z}}{z} \rho \bar{v} - \frac{1}{a} \bar{P} \bar{\nabla} \rho & \text{momentum "} \\ \frac{\delta \bar{E}}{\delta t} + \frac{1}{a} [(\bar{E} + \bar{P}) \bar{v}] = -2 \frac{\dot{z}}{z} \bar{E} - \frac{1}{a} \rho \bar{v} \bar{\nabla} \bar{P} & \text{energy "} \end{cases}$$

- They are a set of conservation laws: $\frac{\delta w}{\delta t}(\bar{x}, t) = -\bar{\nabla} \bar{F}(\bar{x}, t)$ (1) $w = \text{field: } \rho, \varepsilon, \dots$
- 2 ways to solve them: A, B $\bar{F} = \text{flux of: } \rho, \varepsilon,$

A) Eulerian approach:

$$\begin{aligned} w(t) &= w(t_0) + \left. \frac{\delta w}{\delta t} \right|_{t_0} (t - t_0) + \dots \\ &\approx w(t_0) - \bar{\nabla} \bar{F} \Big|_{t_0} (t - t_0) \\ &\approx w(t_0) - \frac{f_{i+1} - f_i}{\Delta x} \Big|_{t_0} (t - t_0) \end{aligned}$$

$\hookrightarrow$ keep 1st order, use (1)
 $\hookrightarrow$ discretize

B) Lagrangian method: Smooth Particle Hydrodynamic (SPH)

- macro extended particles with: $\bar{x}_i, \bar{v}_i, \bar{\rho}_i, \varepsilon_i$ (energy density)
- solve the flux equations at the position of each particle

$$\begin{aligned} \langle f(\bar{x}) \rangle &= \int f(\bar{x}') \mathcal{K}_h(\bar{x} - \bar{x}') d^3 \bar{x}' \\ &\approx \sum_i^N f(\bar{x}_i) \mathcal{K}_h(\bar{x} - \bar{x}_i) \left(\frac{m_i}{\bar{\rho}_i} \right)^{dV} \end{aligned}$$

$\mathcal{K}_h(\bar{x} - \bar{x}')$ smoothing kernel
 $h = \text{smoothing scale}$
 dV : mass resolution is fixed

i.e. approximate field with a spline
 $\leftrightarrow$ superposition of all particles

Density field (example)

$$\langle \rho(\bar{x}) \rangle = \sum_i \frac{m_i}{\rho_i} \rho_i \rho_h(\bar{x} - \bar{x}_i) = \sum_i m_i \rho_h(\bar{x} - \bar{x}_i) \quad w = \rho$$

$$\frac{\delta w}{\delta t}(\bar{x}, t) = -\bar{\nabla} \bar{F}(\bar{x}, t)$$

Derivatives

$$\delta_j \langle \bar{f}(\bar{x}) \rangle = \sum_i \frac{m_i}{\rho_i} \bar{f}(\bar{x}_i) \delta_j \rho_h(\bar{x} - \bar{x}_i) \quad j\text{-th direction}$$

symmetric representation of derivative to reduce numerical errors: (numerical trick)

$$\frac{\bar{\nabla} \bar{f}}{\bar{\rho}} = \bar{\nabla} \left(\frac{\bar{f}}{\bar{\rho}} \right) + \frac{\bar{f}}{\bar{\rho}^2} \bar{\nabla} \bar{\rho} \quad \bar{\nabla} \bar{f} = \bar{\rho} \bar{\nabla} \left(\frac{\bar{f}}{\bar{\rho}} \right) + \frac{\bar{f}}{\bar{\rho}} \bar{\nabla} \bar{\rho} \quad (\text{product rule of derivatives})$$

$$\Rightarrow \bar{\nabla} \langle \bar{f}_j \rangle = \bar{\rho}_j \sum_i \frac{m_i}{\rho_i^2} \bar{f}(\bar{x}_i) \bar{\nabla} \rho_h(\bar{x} - \bar{x}_i) + \frac{\bar{f}_j}{\bar{\rho}_j} \sum_i m_i \bar{\nabla} \rho_h(\bar{x} - \bar{x}_i)$$

$$= \sum_i m_i \left(\frac{\bar{f}_i}{\rho_i^2} + \frac{\bar{f}_j}{\rho_j^2} \right) \bar{\nabla} \rho_h(\bar{x} - \bar{x}_i)$$

Include presence of shocks

- Shocks are not captured by hydro eq.s
- Include artificial viscosity, Π_{ij} , to transfer kinetic energy in to heat i.e. coherent velocities $\rightarrow$ random (thermal)
- (in shocks energy $\uparrow$ but we assume $S = \text{const.} \Rightarrow \Pi_{ij}$ allows $S \uparrow$)

Smooth Particle Hydrodynamics (SPH)

$$\left\{ \begin{aligned} \frac{d\bar{v}_j}{dt} &= - \sum_i m_i \left(\frac{\bar{p}_i}{\bar{\rho}_i^2} + \frac{\bar{p}_j}{\bar{\rho}_j^2} + \Pi_{ij} \right) \bar{\nabla} \rho_h(\bar{x}_j - \bar{x}_i) \end{aligned} \right.$$

Euler

$$\left\{ \begin{aligned} \frac{d\bar{\varepsilon}_j}{dt} &= \frac{1}{2} \sum_i m_i \left(\frac{\bar{p}_i}{\bar{\rho}_i^2} + \frac{\bar{p}_j}{\bar{\rho}_j^2} + \Pi_{ij} \right) \bar{\nabla} \rho_h(\bar{x}_j - \bar{x}_i) \cdot (\bar{v}_j - \bar{v}_i) \end{aligned} \right.$$

Energy

continuity eq. is automatically satisfied by particles conservation
 $\Rightarrow$ no need to evaluate it

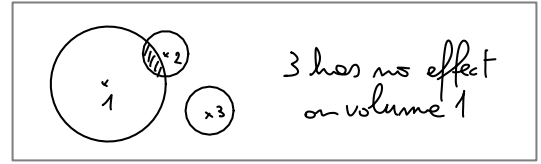
Continuity

- Numerical issues with Kelvin-Helmholtz instability

Kernel

• Function with Compact support $k(\bar{x})=0$ for $|\bar{x}|>h$

$$\text{eg. } k_h(x) = \frac{8}{\pi h^3} \begin{cases} 1 - 6\left(\frac{x}{h}\right)^2 + 6\left(\frac{x}{h}\right)^3 & 0 \leq \frac{x}{h} < 0,5 \\ 2\left(1 - \frac{x}{h}\right)^3 & 0,5 \leq \frac{x}{h} < 1 \\ 0 & 1 \leq x/h \end{cases}$$



$\Rightarrow$ Sum over neighbours (in a sphere of radius h), not over the entire volume

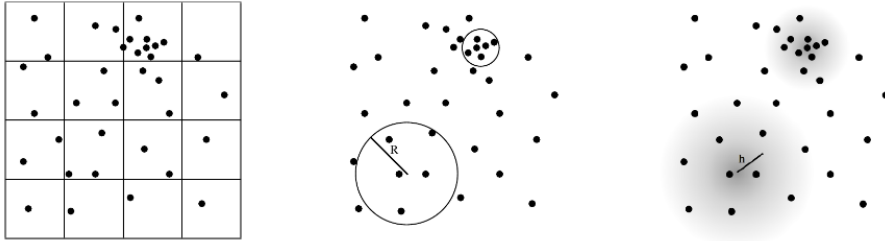


Figure 4.2.: Different methods to compute a continuous density from point mass particles are: the particle-mesh method (left), constructing a local volume according to the local density of particles (middle), and the approach adopted in smoothed particle hydrodynamics (right).

Image credit: [Price \(2012b\)](#)

"Other" ways to solve hydro equations

eg. Mesh, Adaptive-mesh
eg. Voronoy tessellization (eg. AREPO code)

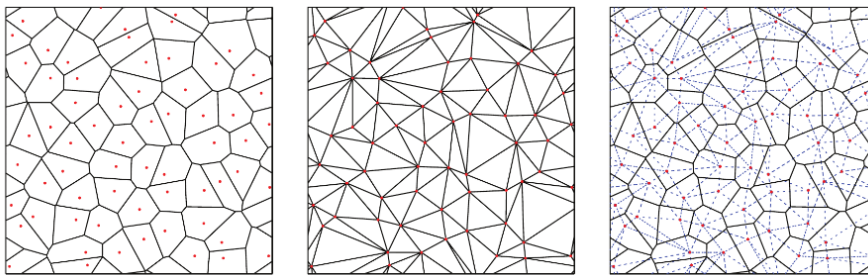
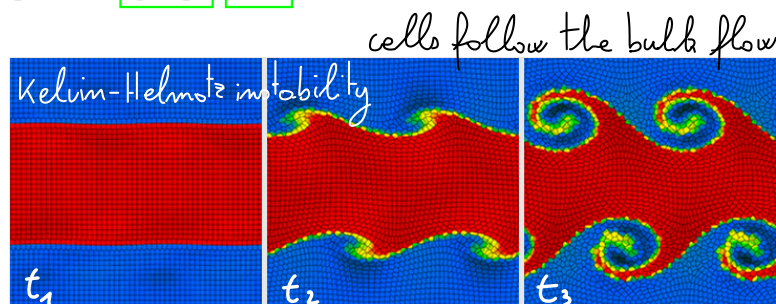


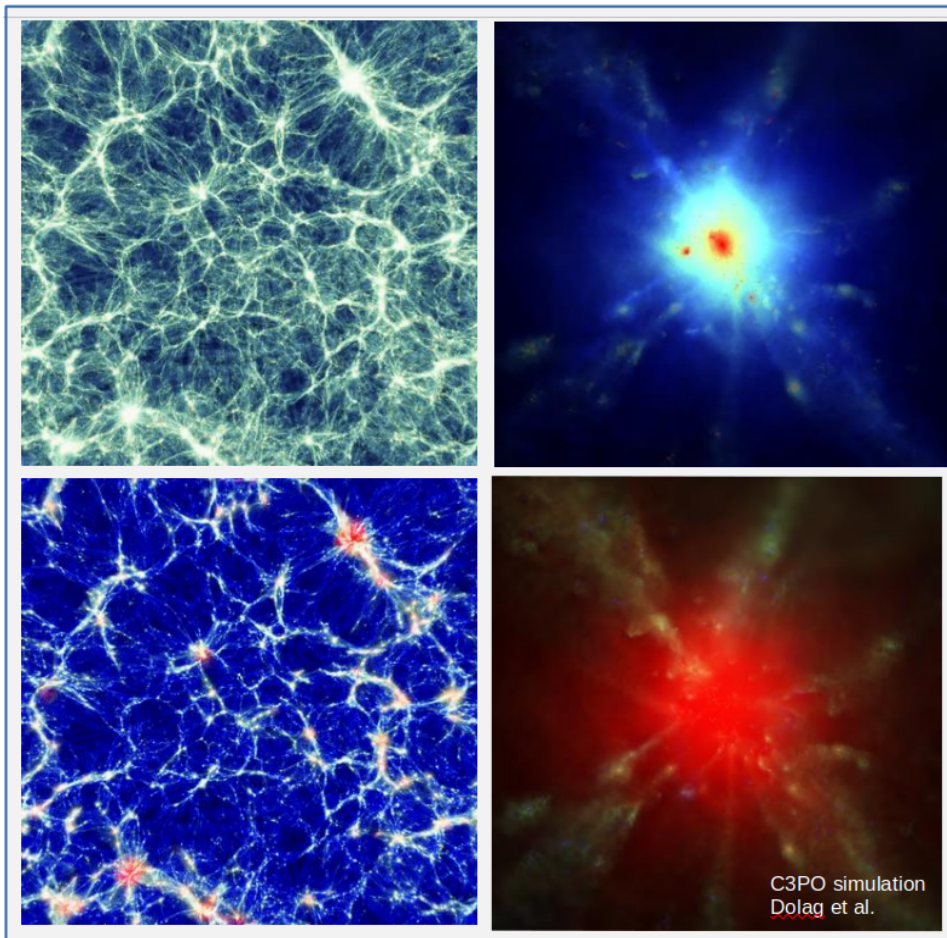
Figure 4.1.: Voronoi (left) and Delaunay (middle) tessellation of a two-dimensional periodic box. The Delaunay tessellation provides a complete set of perpendicular bisectors needed for the construction of the irregular moving-mesh.

Image credit: [Springel \(2010a\)](#)



cells follow the bulk flow

Very realistic representation of the data



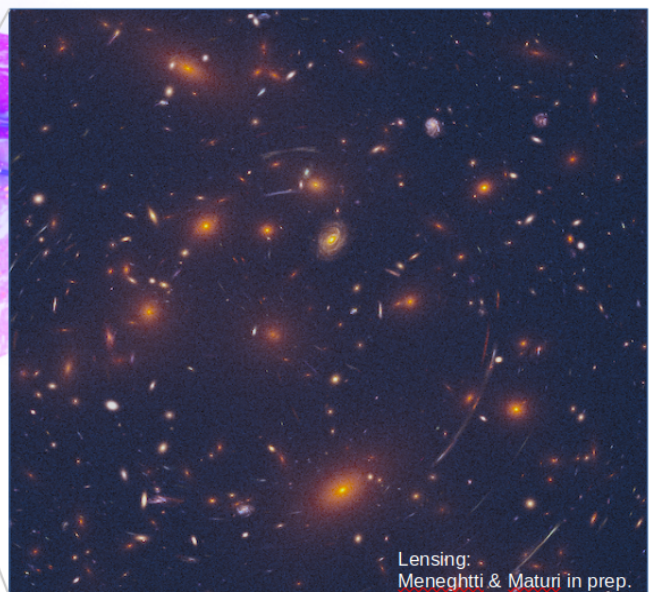
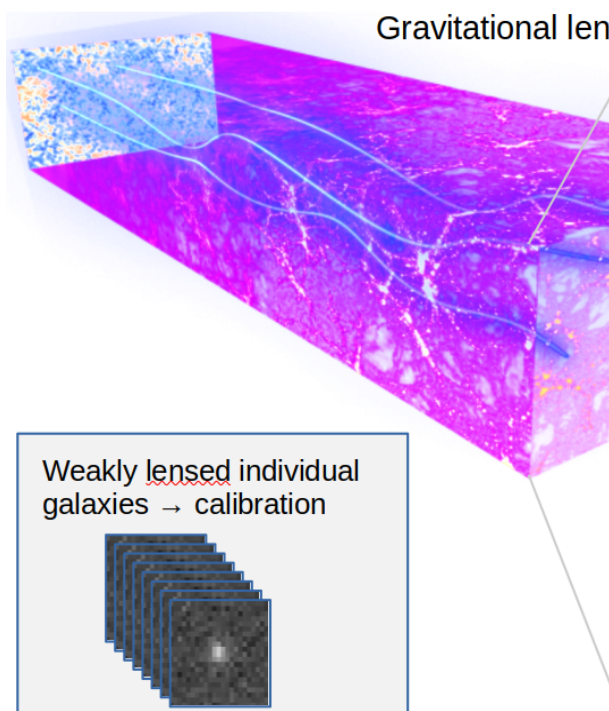
Observables (1)

- density
- temperature
- turbulence
- gravitational lensing
- magnetic fields
- ...

Observables (2)

Ray tracing simulations

Gravitational lensing

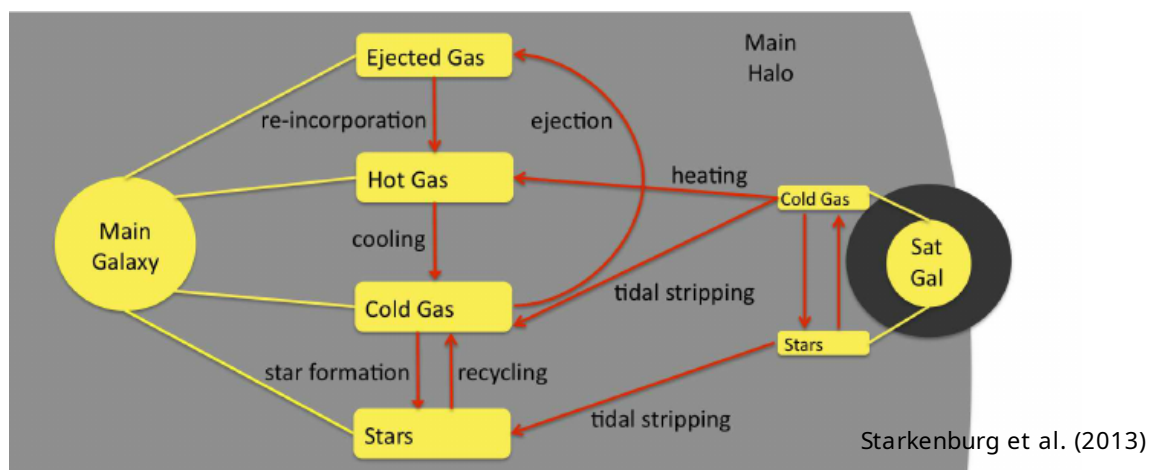
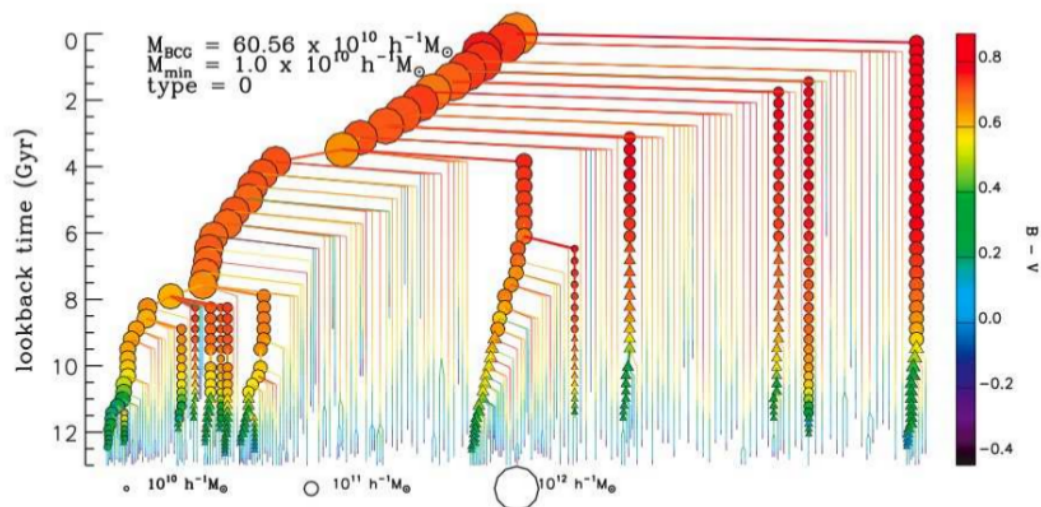


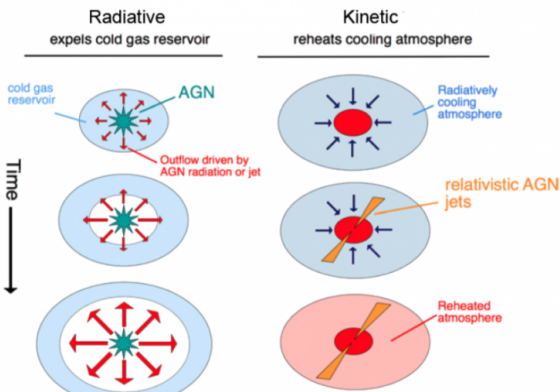
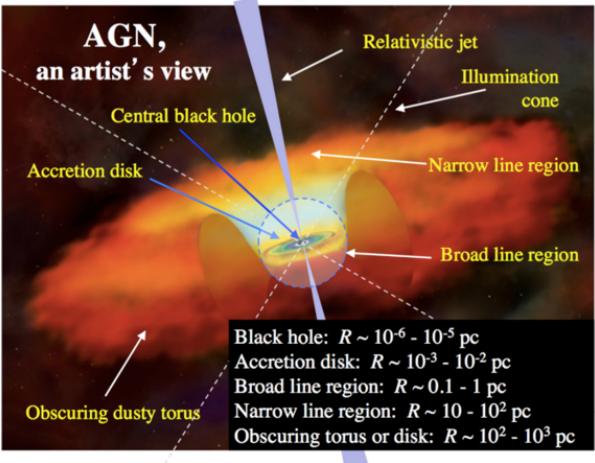
Semi-Analytical models

- Baryons give rise to further complex physics
- given certain local density, $\rho(\vec{x}, t)$, pressure, $P(\vec{x}, t)$, temperature, $T(\vec{x}, t)$ (from hydro eq.s) baryons can "collapse in galaxies" $\rightarrow$ Stars, AGNs, SN,
- Add sub-grid physics: SN, AGN feedback, chemistry, shocks,
 - $\hookrightarrow$ they are a set of analytical expressions to account for that
 - $\hookrightarrow$ for example, include formation of AGNs $\Rightarrow$ include energy injection in the medium deLucia et al.

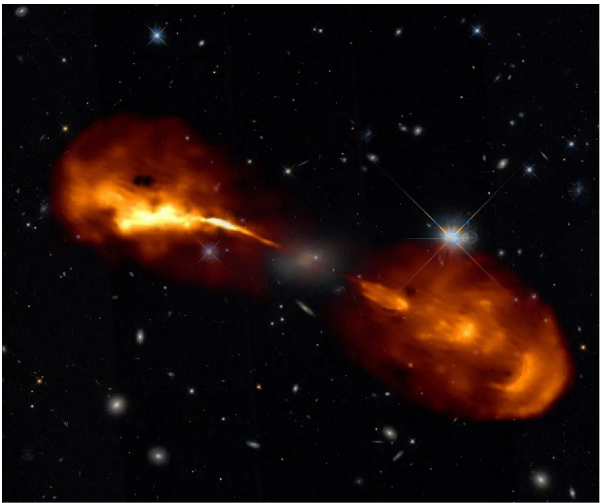
Semi-analytical models: a sketch

- Each halo has a **mass** and a **position**
- Accordingly attach a physical/phenomenological recipe
- Follow the story of each haloe (Milky-satellites)





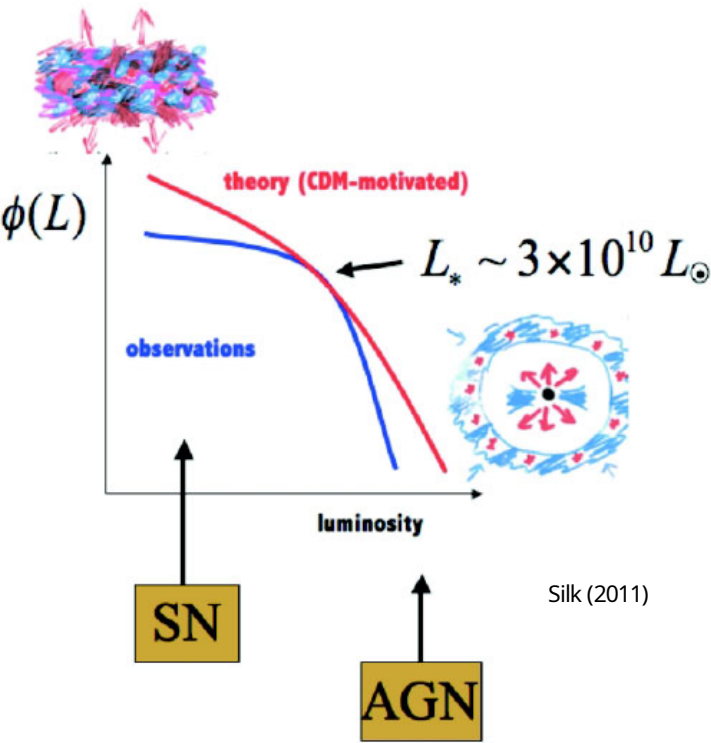
Fabian (2016)



AGN feed back
Hercules A (R. Timmerman; LOFAR & Hubble Space Telescope)



Ram pressur stripping
ESO 137-001 (NASA/ESA/Hubble Heritage Team)

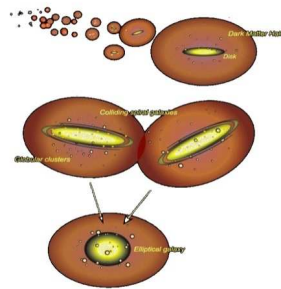


Merger trees and semi-analytic modelling

Matteo Maturi

Heidelberg University

Cosmology WS 2025/26



Summary

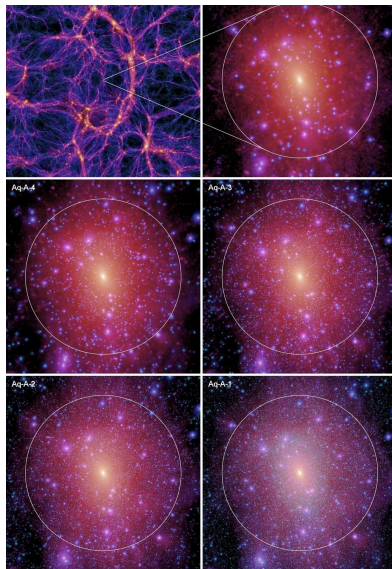
Subject

- The satellites of the Milky Way - Insights from semi-analytic modelling in a LambdaCDM cosmology
Starkenburg E., Helmi A., De Lucia G., Li Y.-S., Navarro J. F., Font A. S., Frenk C. S., Springel V., Vera-Ciro C. A. and White S. D. M.
- The Spatial Distribution of Galactic Satellites in the Λ CDM Cosmology
Wang J., Frenk C. S. and Cooper A. P.

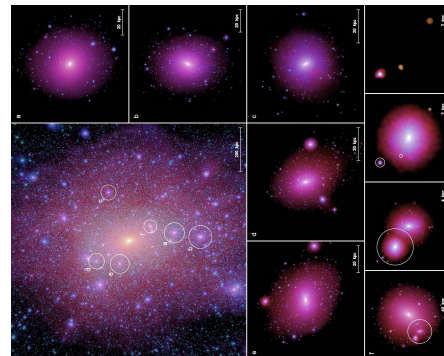
Content

- The satellites: Where? Star content? Metallicity properties? Comparison with hydrosim.
- Aquarius simulation
- Intro to semianalytical models
- Papers semi-analytical model
- Papers results

Aquarius simulations



resimulations

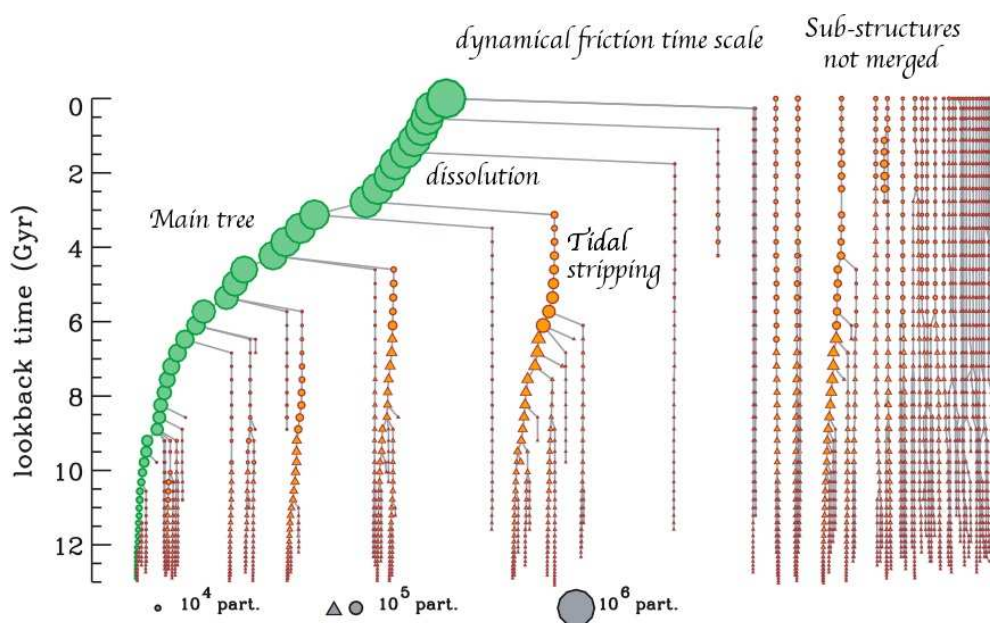


substructures in substructures

- DM only simulation (cube $100h^{-1}Mpc$) ($\Rightarrow$ full LCDM context)
- Resimulation of 6 isolated Milky-like galaxies
- Different resolutions up to about 200 million particles
- One halo at 1.5 billion particles (IGNORED HERE).

The merger tree: DM

1. **Identify** haloes down to 20 particles with FoF
2. Follow the mergin history of haloes $\Rightarrow$ **Merging tree**
3. Identify the **'first' progenitor**: $M_i = N_i + \max(M_{i_1}, M_{i_2}, \dots, M_{i_n})$,
(N_i number of self-bound particles of halo i, M_{i_j} progenitor of the halo i)



Physics between haloes

Gravitational interactions between haloes

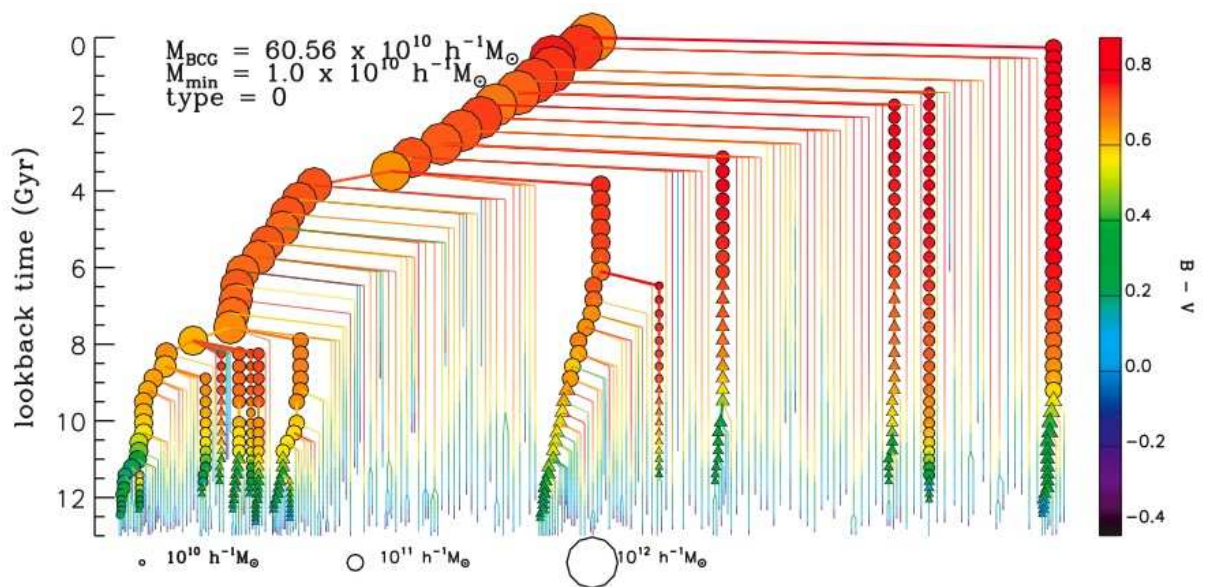
- Merging of haloes
- Tidal truncation, tidal stripping (merging haloes lose pieces)
- Dynamical friction (merging haloes slow down and get absorbed)
- Gravitational dissolution (haloes get smaller than 20 part.)
- Physics within haloes (sub-grid)...



Here it starts the spherical-cow domain

Semi-analytical models: a sketch

- Each halo has a **mass** and a **position**
- Accordingly attach a physical/phenomenological recipe
- Follow the story of each haloe (Milky-satellites)



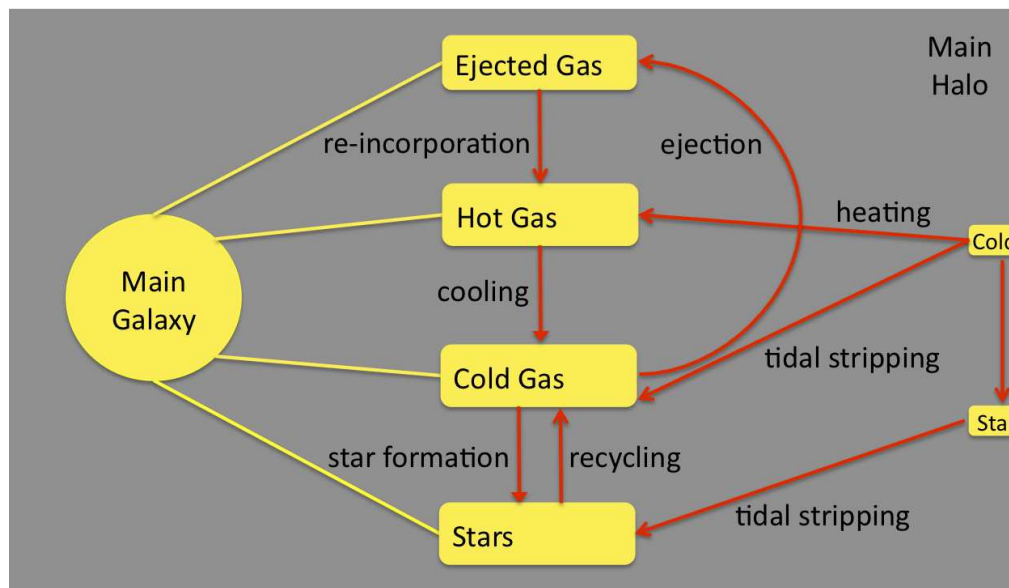
Paper Semi-analytic model

- **Reionization**
 - it reduces the content of barions in haloes (filtering mass)
 - Use global reionization ($15 > z > 11.5$) stronger than usual
 - Anyway local reionization by local sources is full at $z = 10$
- **Cooling**
 - depends on haloes metallicity and temperature
 - no cooling below $10^4 K$
 - helps star formation
- **Star formation**
 - cool gas in discs above a certain mass goes into stars
 - star formation is proportional to that amount of gas
- **Passive stellar evolution**
 - IMF: what is the mass distribution of stars when they form
 - Stellar evolutionary tracks (Padova 1994)
- **SN feedback**
 - Gas heating and ejection of gas
 - fraction of ejected gas depends on the halo potential ($\propto V_{200}^{-2}$)
 - gas can be reaquired

Semianalytical model

- Metals
 - NO full chemical chain
 - follows the star formation events
 - fine only for Type-II SN
- If haloe turns into a satellite
 - Ram pressure stripping of hot gas component
 - Ejected gas component
 - Star stripping: stars goes to central galaxy
do not affect luminosity function, because a given star remains so
- Tidal stripping → goes to central galaxy
 - when particles of one haloe get < 20 particles they are followed by their (sub bixel) recipe (orphans)
 - important for luminosity function
 - BUT: how do they follow their spatial distribution???

A more detailed sketch



Part V

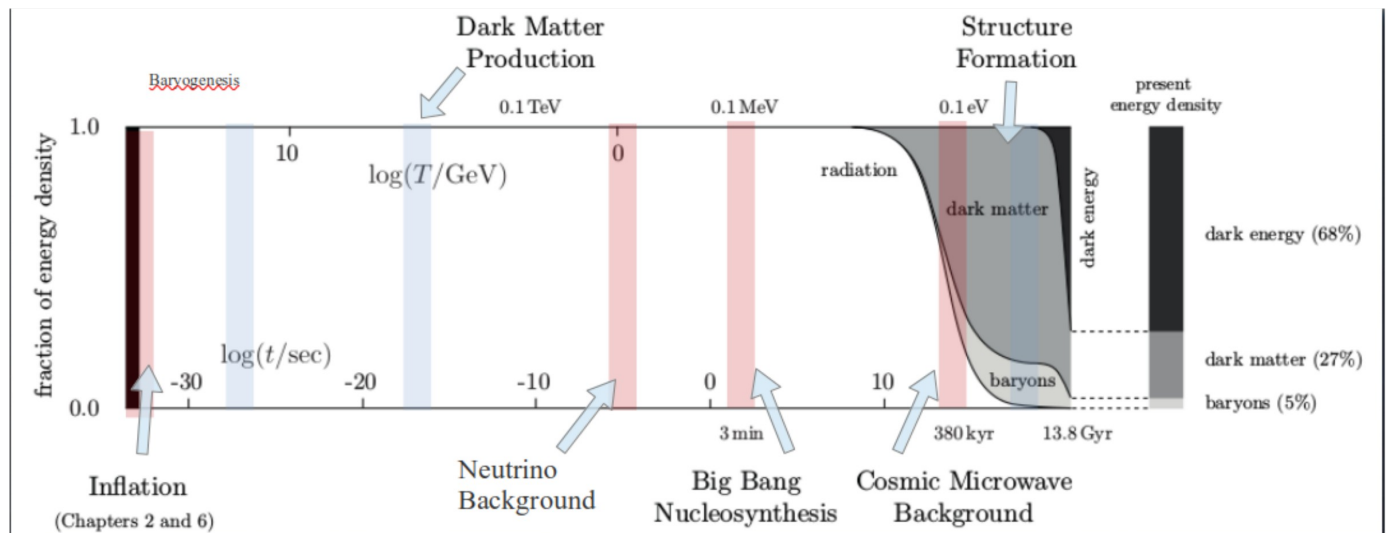
Observations and cosmological probes

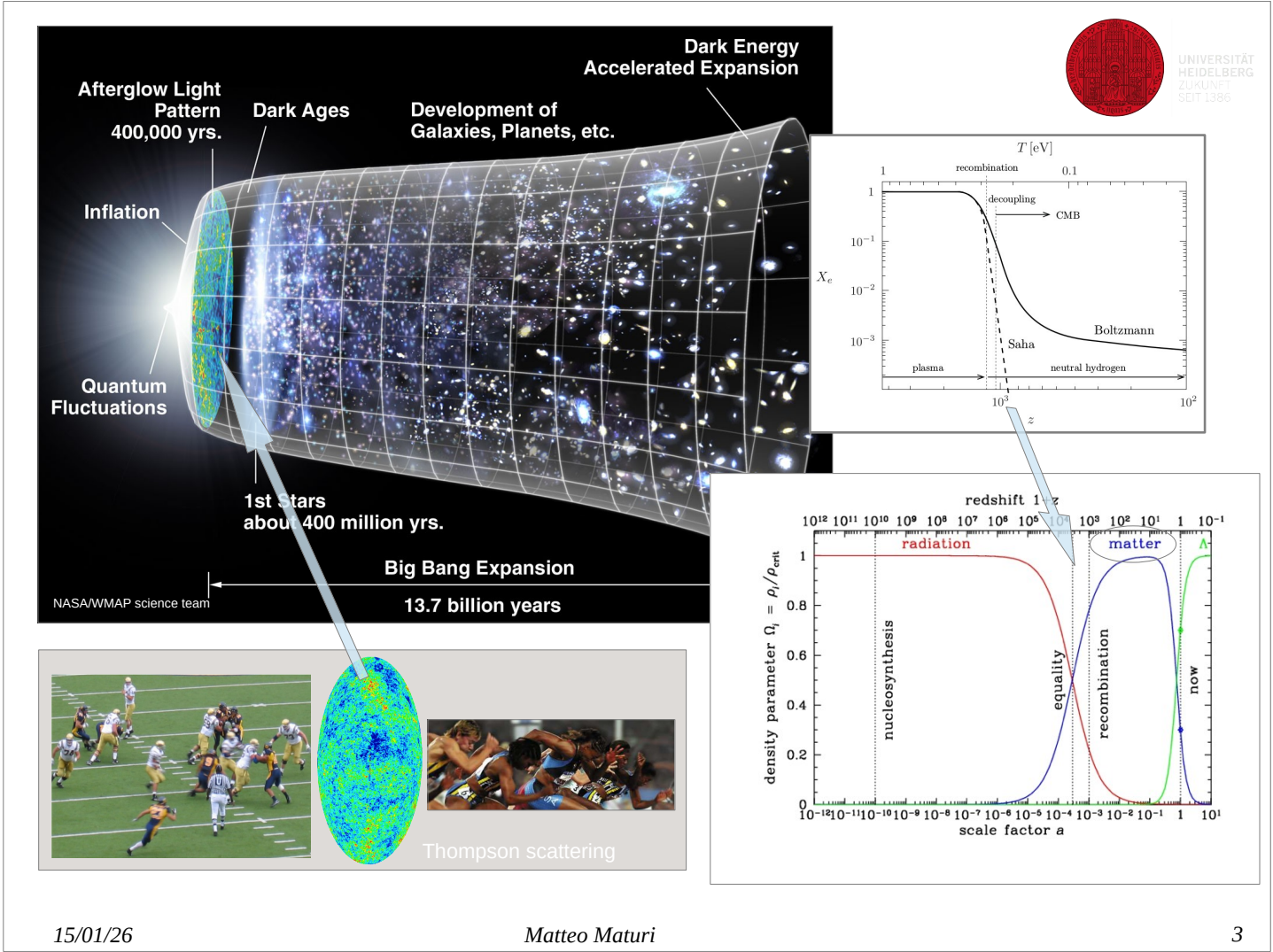


Cosmic Microwave Background: intro and the power spectrum

Matteo Maturi

Heidelberg University





15/01/26

Matteo Maturi

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Few simple facts



Some history:

- Last scattering surface => $z=1100$
- Up to recombination baryons and photons are tightly coupled => same T

$a(1+z)^{-1}$	=> universe is 1000 smaller
$T_r \propto a^{-1} = (1+z)$	=> " " 1000 hotter
$T \propto a^{3/2} = (1+z)^{-3/2}$	=> " " photons traveled for 99,997% age of the universe
$\rho \propto_0 0.04(1+z)^3$	=> " " 10^9 times denser

- Linear perturbations (analytical)
- Simple physics (adiabatic contractions) | => It is a clean observable = **Powerful!**
- Strong prove for Dark Matter:

Amplitude with linear growth from $a=1$ backward, fluctuations turn out as: $\frac{\delta(a=1)}{D_+(a_{CMB})} \geq a_{CMB}^{-1} \approx 10^{-3}$
 (flat)

But in reality they are $\delta(a_{CMB}) \sim 10^{-5}$. Why?

- DM feels only gravity
- => δ_{DM} continue growth ($p=0 \rightarrow$ no oscillations)
- => the estimate does not account for this
- => structures form "more" and when we rewind we start from a $\delta(a=1)$ which is higher

- Important scale : $R_h \sim 1.7^\circ$ how can it be so uniform?
- $\langle T \rangle = 2.725K$ => $\pi \approx 1mm$

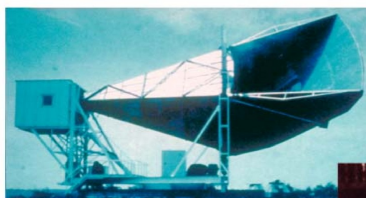
The discovery of the CMB



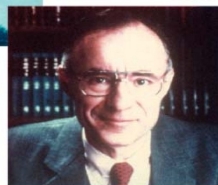
Some history:

- **1946** Robert Dicke. Radiation of cosmic matter, predic: $T < 20\text{K}$ (not yet CMB)
- **1946** George Gamow. Predict $T \approx 50\text{K}$ assuming $H = 3 \cdot 10^6$ years ant actual temperature of IGM
- **1948** Ralph Alpher & Robert Herman. Estimate $T \approx 5\text{K}$
- **1949** Ralph Alpher & Robert Herman. Correct to $T \approx 28\text{K}$
- **1953** George Gamow. Estimate $T \approx 7\text{K}$.
- **1956** George Gamow. Estimate $T \approx 6\text{K}$.
- **1960s** Robert Dicke. Estimate $T \approx 40\text{K}$ name it MBR (Microwave Background Radiation)!
- **1965** Arno Penzias & Robert Woodrow Wilson. Measure $T \approx 3\text{K}$ as BigBang signature (CMB)!

“Fixing antennas”...



Microwave Receiver



Arno Penzias

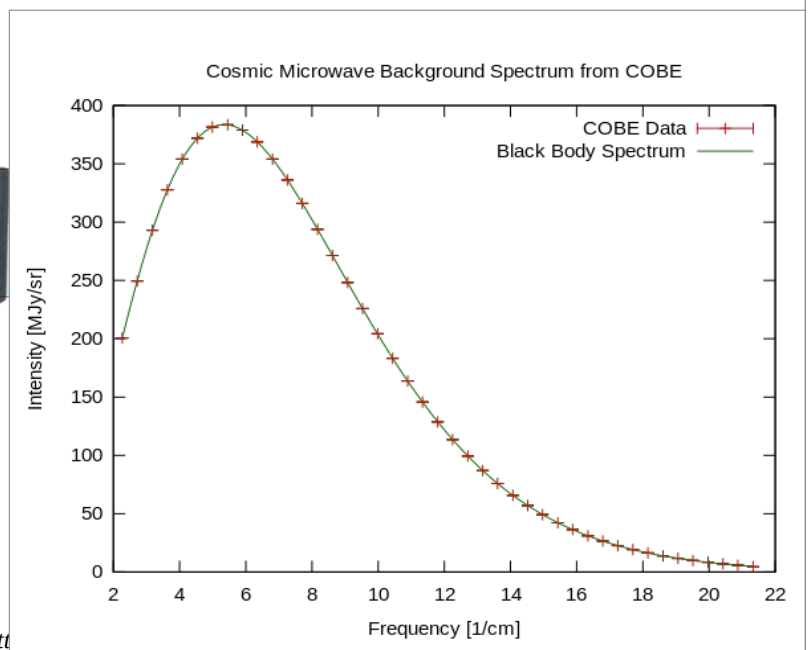


Robert Wilson

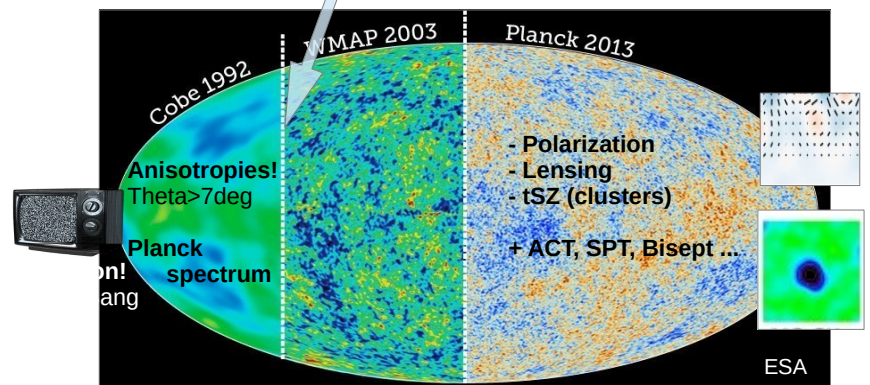
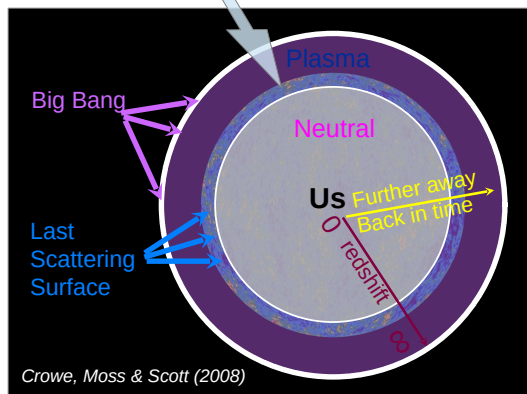
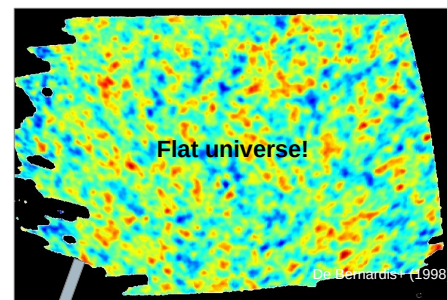
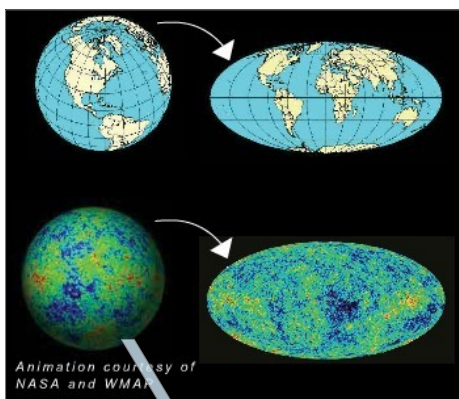
MAP990045

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Matt



CMB observations

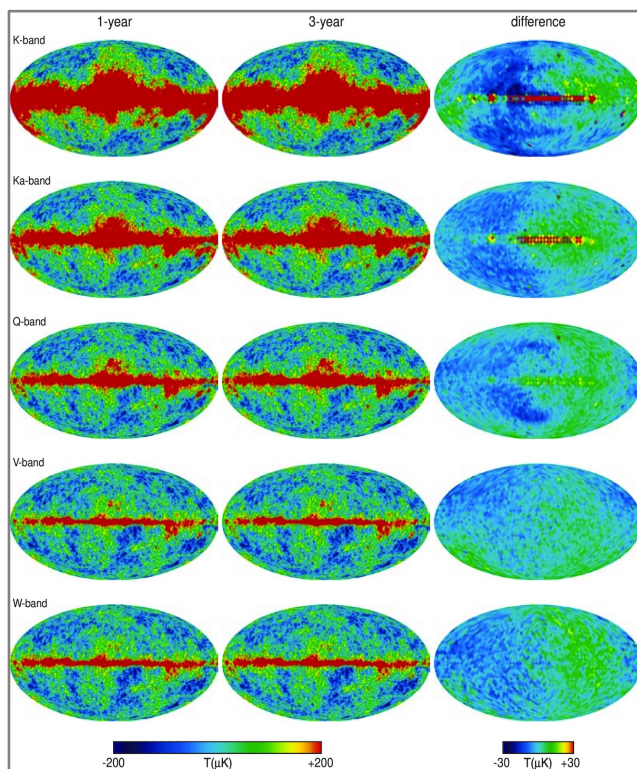


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Not easy... Galactic foreground

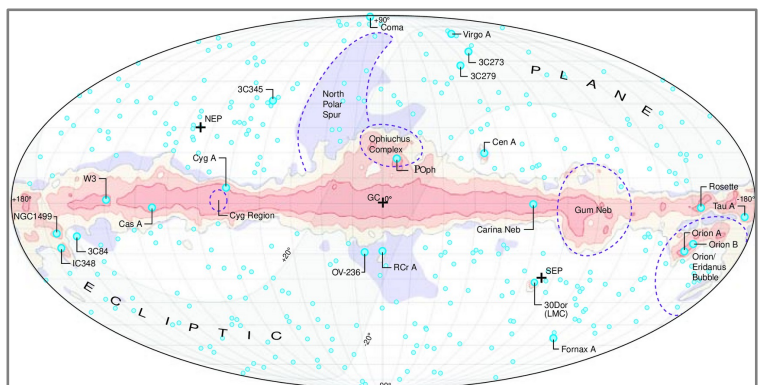


Objects:

- gas
- dust
- localized sources:
SN remnants, star forming regions...

Process:

- synchrotron
- free-free
- thermal (black body)

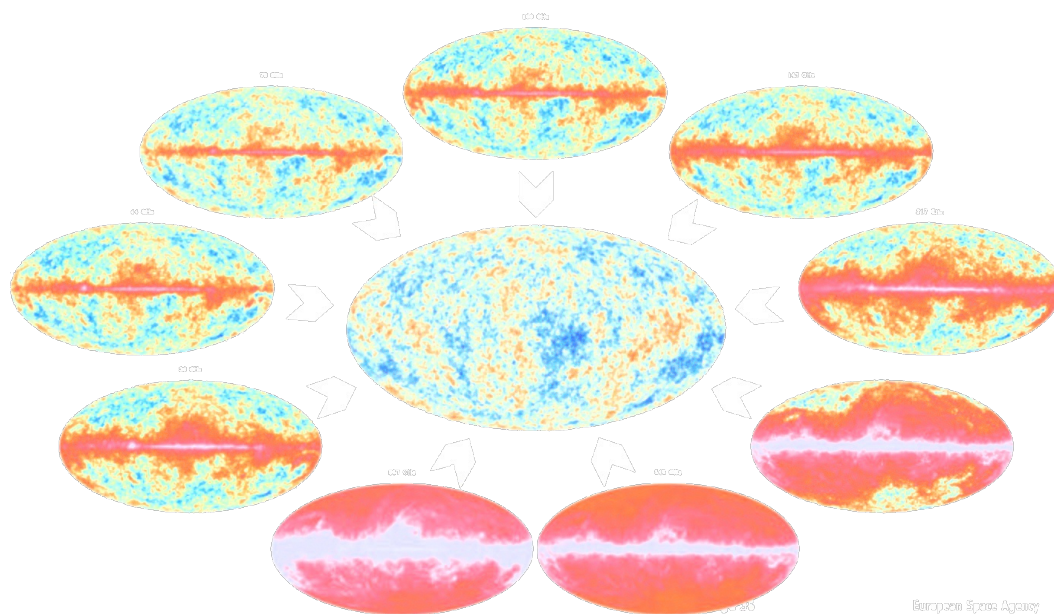


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But possible... Components separation



European Space Agency

$$\delta T_{R-J}(\nu) = T_{sync} \left(\frac{\nu}{\nu_K} \right)^{\beta_{sync}(\nu)} + T_{free} \left(\frac{\nu}{\nu_K} \right)^{\beta_{free}(\nu)} + \delta T_{CMB} a(\nu) + T_{dust} \left(\frac{\nu}{\nu_K} \right)^{\beta_{dust}(\nu)}$$

Dealing with a sphere: Decomposition in Spherical harmonics



We see Temperature fluctuations projected on a 2D surface (sky)

$$\Theta(\hat{\mathbf{n}}) = \frac{T(\hat{\mathbf{n}}) - \langle T \rangle}{\langle T \rangle} = \frac{\delta T}{T}(\hat{\mathbf{n}}) = \sum_{lm} a_{lm} Y_{lm}(\hat{\mathbf{n}}) \quad \text{Temperature fluctuations}$$

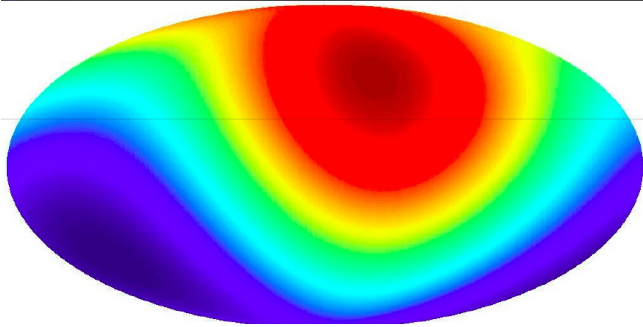
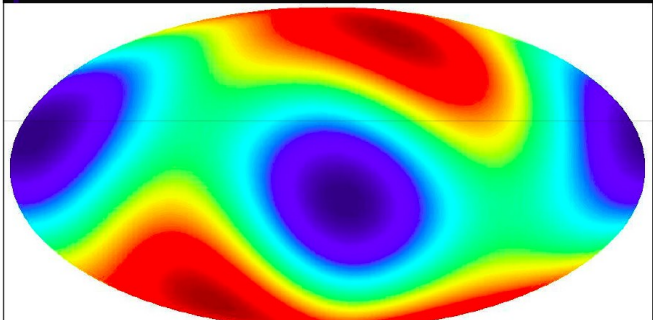
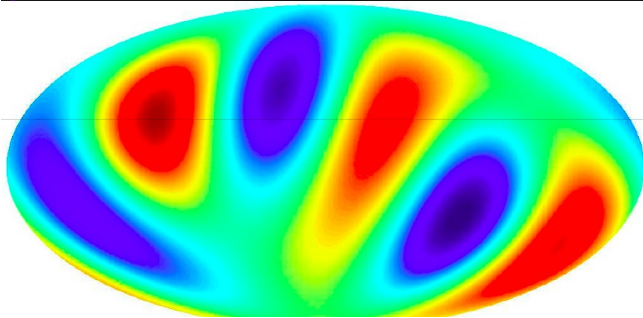
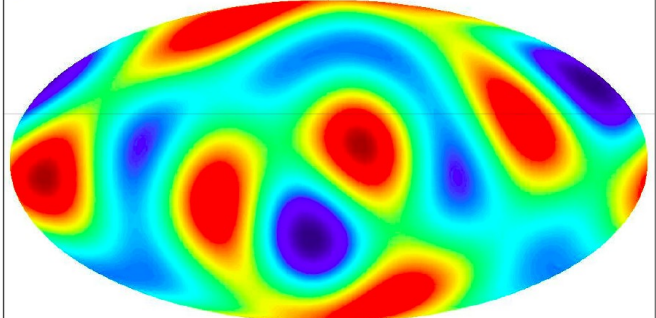
$$Y_{lm} = \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} P_{\ell}^m(\cos\theta) e^{im\phi} \quad \begin{array}{l} \text{Spherical harmonics} \\ P_{\ell}^m \text{ are the Legendre polynomials} \end{array}$$

$$a_{lm} = \int_{\theta=-\pi}^{\pi} \int_{\phi=0}^{2\pi} \Theta(\hat{\mathbf{n}}) Y_{lm}^*(\hat{\mathbf{n}}) d\Omega \quad \text{Coefficients}$$

$$\langle a_{lm} a_{l'm'}^* \rangle = \delta_{\ell\ell'} \delta_{mm'} C_{\ell} \quad \text{Power spectrum}$$

ℓ is called 'multipole' and represents a given angular scale in the sky $\theta \sim \pi/\ell$

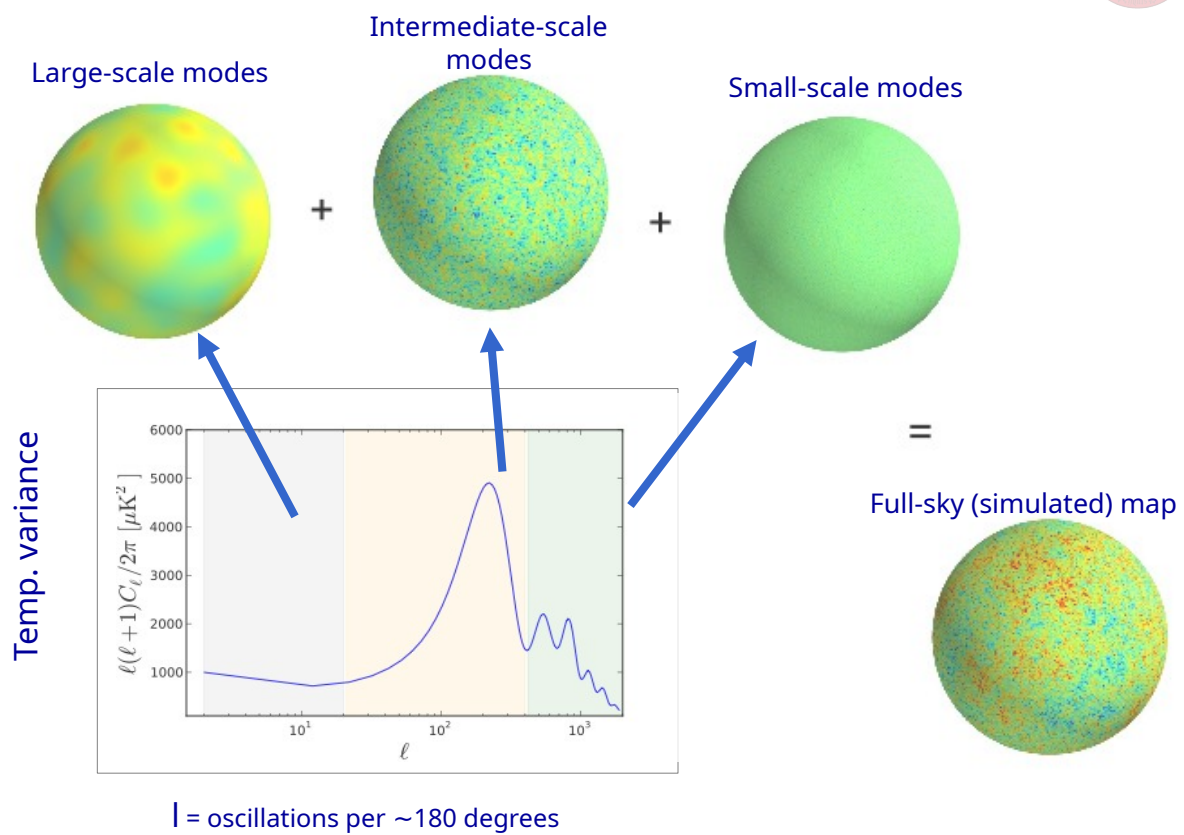
How the modes look like?

 $\ell=1$  $\ell=2$  $\ell=3$  $\ell=4$ 

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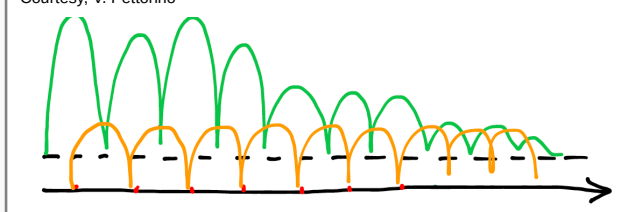
Courtesy of Douglas Scott

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Courtesy, V. Pettorino



Power spectrum

Compression and expansion ($\lambda > R_H$)

- Velocity $\rightarrow$ Doppler-effect
- Gravitational t. delay
- Gravitational redshift $\rightarrow$ SW effect

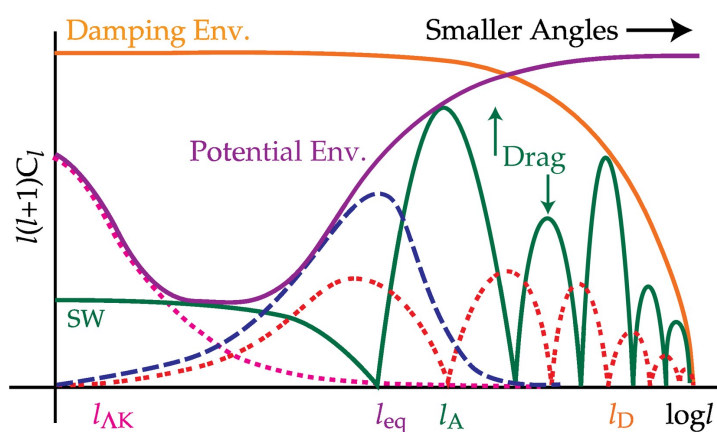
Harmonic oscillations ($\lambda < R_H$)

- pressure is active
- velocity and compression

Baryon dragging

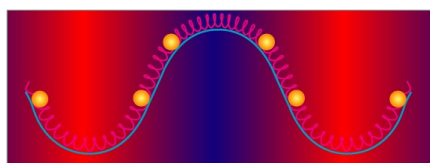
Damping suppression

Velocity and density add in quadrature



- Late ISW
- Early ISW
- Eff. Temp.
- Doppler

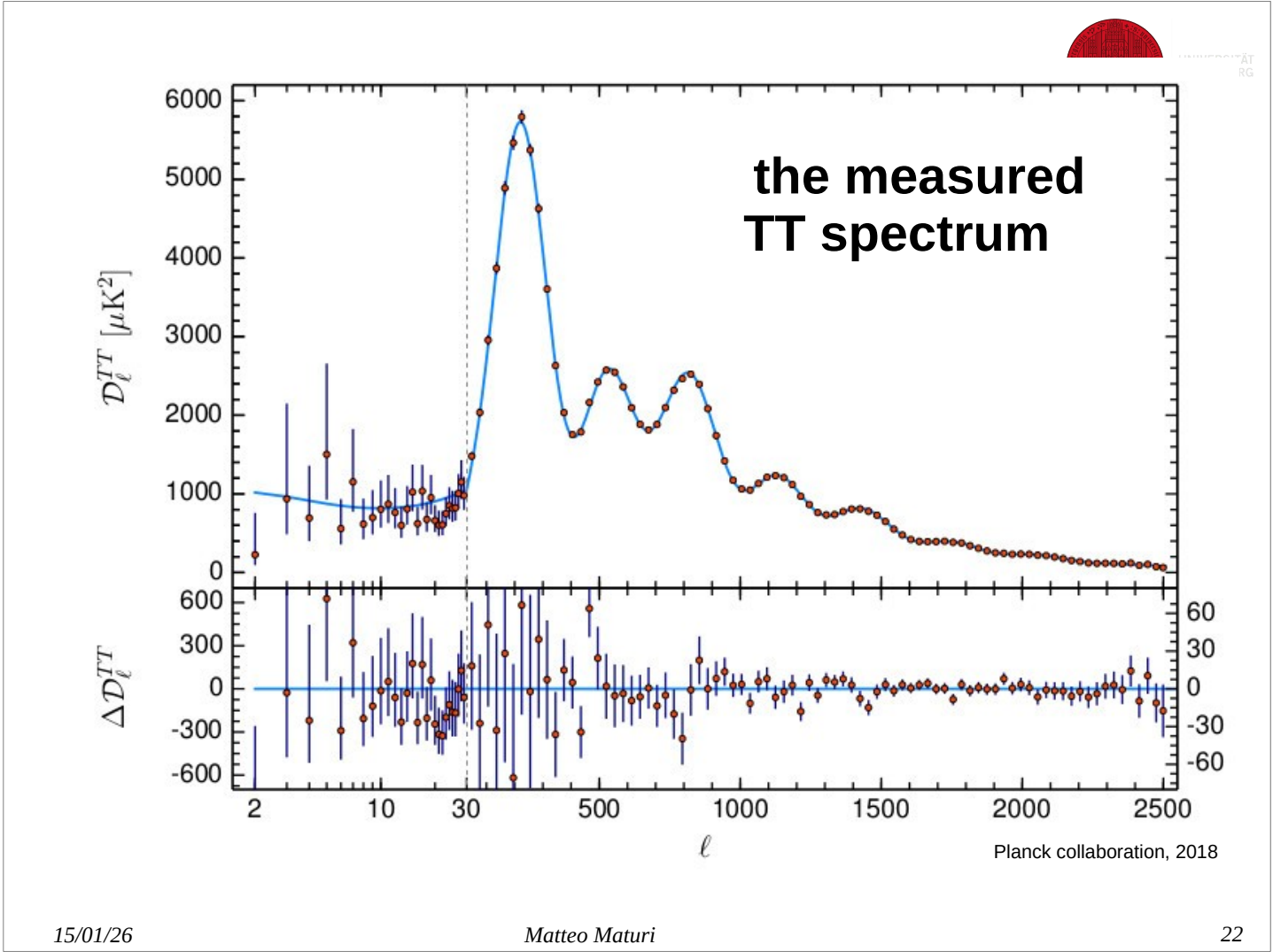
Hu (1995)



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CMB temperature fluctuations

▫ Observable : $T_0 = \langle T(\bar{\theta}) \rangle = 2,725\text{K}$ average temperature
 $\delta T(\bar{\theta}) = T(\bar{\theta}) - T_0$ fluctuations $\bar{\theta} = \{\text{R.A., Dec.}\}$

▫ Fields coupled to photons

- 1) Gravity : $\varphi(\bar{r})$ gravitational blue/red-shift, time delays
- 2) Density : $\delta(\bar{r})$ adiabatic expansion/compression
- 3) Velocity : $\bar{v}(\bar{r})$ doppler effect

▫ Conceptually

$$\frac{\Delta T}{T}(\bar{\theta}, \varphi) = \underbrace{\frac{\varphi(\bar{r})}{3c^2}}_{\text{gravity}} + \underbrace{\frac{1}{3}\delta(\bar{r})}_{\text{density}} - \underbrace{\frac{\bar{v}(\bar{r})}{c}}_{\text{velocity}} + \underbrace{\frac{2}{c^2} \int \dot{\varphi}(\bar{r}, t) dt}_{\varphi \text{ evolution}} + \underbrace{tS_z + kS_z}_{\text{galaxy clusters}} + \underbrace{\text{lensing + Reionization}}_{\text{LSS}}$$

adiabatic comp. $T \uparrow$?

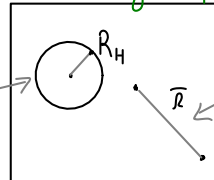
Convolution

▫ Anisotropies
 primary $z \gg z_{\text{rec}}$ info on primordial conditions
 secondary $z < z_{\text{rec}}$ " " structures

▫ 2 Regimes :

$r < R_H(z_{\text{CMB}}) \Rightarrow P \neq 0$
Acoustic Oscillations

last scattering surface



$r > R_H(z_{\text{dec}}) \Rightarrow P = 0$
 gravity only, No oscillations

(1) $r > R_H(z_{\text{CMB}})$: Gravity

(earth)

- $\bar{v}$: $\bar{v}(\bar{r}) = \bar{v}_0 + \bar{v}_{\text{dec}}(\bar{r})$ $\bar{v}_0$ not interesting $\bar{v}_{\text{dec}} \approx 0$ for $r > R_H$

- φ :
 Grav. Redshift $\frac{\delta T}{T} = \frac{\delta \varphi}{c^2}$ $\delta \varphi < 0$ (overdensity) cooler (red-shift)
 $\delta \varphi > 0$ (underdensity) hotter (blue-shift)

 Time delay $\frac{\delta t}{t} = -\frac{\delta \varphi}{c^2}$ $(\partial \Delta T)_{\text{photons}}$ $\frac{\delta T}{T} = \frac{\delta \Delta}{\Delta} = \frac{2}{3} t^{1/3} t^{2/3} \delta t = \frac{2}{3} \frac{\delta t}{t} = -\frac{2}{3} \frac{\delta \varphi}{c^2}$
radiation domination $\Delta \propto t^{2/3}$

$\Rightarrow$ Sachs-Wolfe (SW) effect

$$\frac{\delta T}{T} = \frac{\delta \varphi}{c^2} - \frac{2}{3} \frac{\delta \varphi}{c^2}$$

$$\boxed{\frac{\delta T}{T} = \frac{\delta \varphi}{3c^2}}$$

(2) $x < R_H(z_{\text{dec}})$: Acoustic oscillations

- photons are leading here, baryons are coupled and follow
DM decoupled, only gravity ψ

• In comoving coordinates (Photon gas, linear regime)

$$\begin{cases} \dot{n} + \bar{n} \bar{u} = 0 & \text{Continuity} \\ \ddot{u} + 2H\dot{u} = -\frac{c^2 \bar{n}}{\bar{\mu} + \bar{p}} \bar{\nabla} \delta p - \bar{\nabla} \delta \psi & \text{Euler} \\ \bar{\nabla}^2 \psi = 4\pi G \bar{\mu} \delta \mu & \text{Poisson} \\ p = \frac{1}{3} \mu c^2 = \frac{\mu}{3} & \text{eq. of state} \end{cases}$$

n = photons number density comoving
 $\mu = \epsilon c^2$ photons energy density
 $p = \frac{\mu}{3}$ photons pressure
 $\bar{u}$ = streaming velocity of the perturbation

• "Ignore" expansion "H=0" + For convenience $z \doteq 1$:(i.e. ignore damping term)
for simplicity

$$\frac{\dot{n}}{\bar{n}} + \bar{n} \bar{\nabla} \bar{u} = 0 \quad \ddot{u} = -\frac{c^2 \bar{n}}{\bar{\mu} + \bar{p}} \bar{\nabla} \delta p - \bar{\nabla} \delta \psi \quad \bar{\nabla}^2 \psi = 4\pi G \delta \mu$$

• Photon gas

$$\square \quad n \propto T^3: \frac{\delta n}{\bar{n}} = 3 \frac{\delta T}{T_0} \quad \square \quad \mu \propto T^4: \frac{\delta \mu}{\bar{\mu}} = 4 \frac{\delta T}{T_0} \quad \square \quad p = \frac{\mu}{3} \propto T^4: \frac{\delta p}{\bar{p}} = 4 \frac{\delta T}{T_0} \quad \square \quad \theta \equiv \frac{\delta T}{T_0}$$

$$\square \quad \mu_0 + p_0 = 4p_0$$

$$3\dot{\theta} + \bar{\nabla} \bar{u} = 0 \quad \ddot{u} = -\frac{c^2 \bar{n}}{4\bar{p}_0} \bar{\nabla} \delta p - \bar{\nabla} \delta \psi = -c^2 \bar{\nabla} \theta - \bar{\nabla} \delta \psi$$

$\dots \rightarrow 3\ddot{\theta} + \bar{\nabla} \ddot{u} = 0 \Rightarrow \boxed{\ddot{\theta} - \frac{c^2}{3} \bar{\nabla}^2 \theta - \frac{1}{3} \bar{\nabla}^2 \delta \psi = 0}$

• To Fourier space + add gravitational redshift by hand

$$\ddot{\theta} + \frac{c^2}{3} k^2 \hat{\theta} + \frac{1}{3} k^2 \hat{\psi} + \frac{\hat{\psi}}{c^2} = 0$$

$\xrightarrow{\text{grav. redshift}} \quad (\bar{\nabla}^2 \rightarrow -k^2) \quad \left(\frac{\hat{\psi}}{c^2} \right)$

• Define $\hat{\theta} \equiv \hat{\theta} + \frac{\hat{\psi}}{c^2}$ effective temperature fluctuations

$$\ddot{\hat{\theta}} + \frac{c^2}{3} k^2 \left(\hat{\theta} - \frac{\hat{\psi}}{c^2} \right) + \frac{1}{3} k^2 \hat{\psi} = 0 \quad \boxed{\ddot{\hat{\theta}} + \frac{c^2}{3} k^2 \hat{\theta} = 0} \Rightarrow \text{for } \dot{\hat{\theta}}_0 = 0 \quad \hat{\theta}(t_{\text{rec}}) = \hat{\theta}_0 \cos\left[\frac{c_s k}{\sqrt{3}} t_{\text{rec}}\right]$$

$R_s = \frac{c_s t_{\text{rec}}}{\sqrt{3}}$ sound horizon at t_{rec}

• At decoupling $p=0 \Rightarrow$ oscillations get frozen

(3) Silk Damping

- free streaming of photons

- Diffusion scale, λ_D , above which the power is damped

$$\lambda \approx (m_e \sigma_T)^{-1}$$

$$dN = m_e \sigma_T c dt = \frac{d\tau}{dt} dt$$

$$\lambda_D \approx \sqrt{N} \lambda$$

$$\lambda_D^2 = \int \lambda^2 \frac{d\tau}{dt} dt = \int \frac{c dt}{m_e \sigma_T}$$

$$C_\ell \rightarrow C_\ell \cdot \exp\left(-\frac{k^2}{k_D^2}\right) \quad k_D \equiv \frac{2\pi}{\lambda_D}$$

mean free path of photons
number of collisions in dt



diffusion scale

diffusion scale

Silk damping: exponential cut off at high ℓ

- small scales baryonic structures are damped

Various contributions to the TT Power Spectrum

Power spectrum

Compression and expansion ($\lambda > R_H$)

- Velocity $\rightarrow$ Doppler effect
- Gravitational t. delay
- Gravitational redshift $>$ SW effect

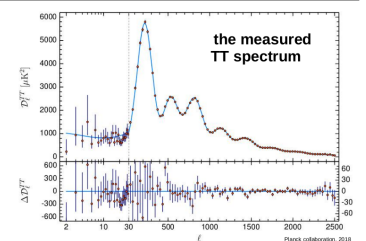
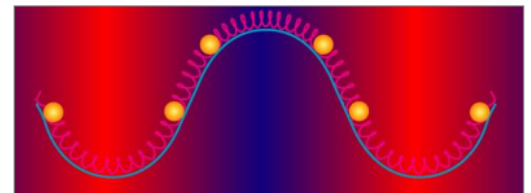
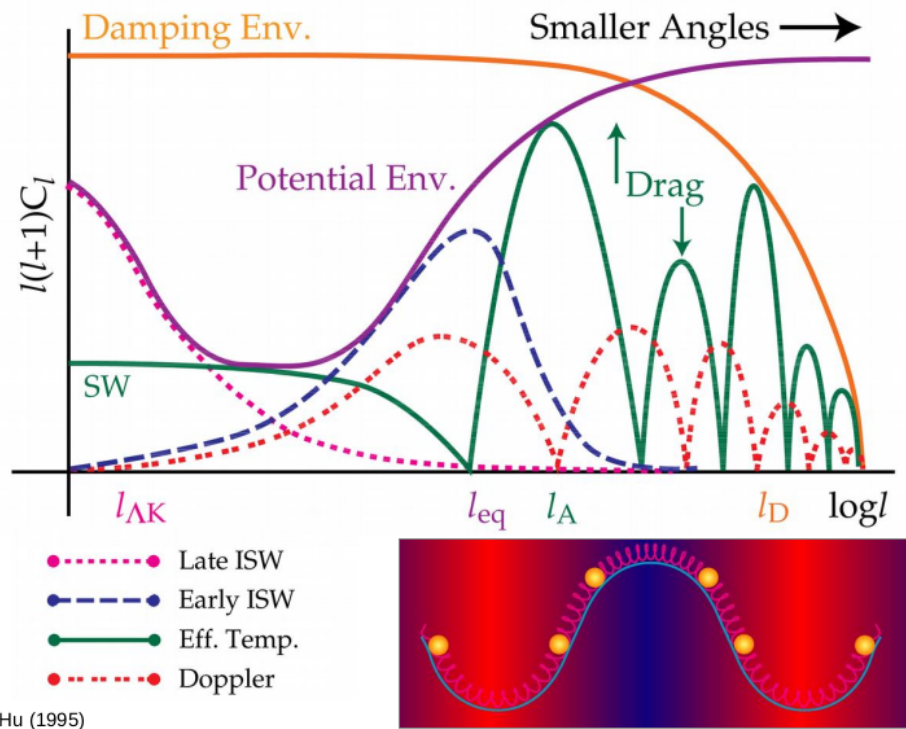
Harmonic oscillations ($\lambda < R_H$)

- pressure is active
- velocity and compression

Baryon dragging

Damping suppression

Velocity and density add in quadrature



DM, B: ρ compression
Y, B: P counter-action } BAO

$c_s = \frac{c}{\sqrt{3}} \approx 0.6c$ sound speed high because $\eta \equiv \frac{M_x}{M_b}$ is very large

$R_s(t) = c_s t$ Sound horizon: defines maximum size of oscillations

(4) Secondary temperature fluctuations $z < 1000$

Integrated Sachs-Wolfe effect (ISW)

$$\frac{\delta T}{\delta t} = \int \dot{\Phi}[\chi(t), t] dt$$

$\downarrow \frac{\delta \Phi}{\delta t}$ conformal time

$\dot{\Phi} = 0$: L.S. $\gamma \rightarrow \gamma'$ $\gamma = \gamma'$ No frequency change

$\dot{\Phi} \neq 0$: L.S. $\gamma \rightarrow \gamma'$ $\gamma \neq \gamma'$ γ' has a different frequency

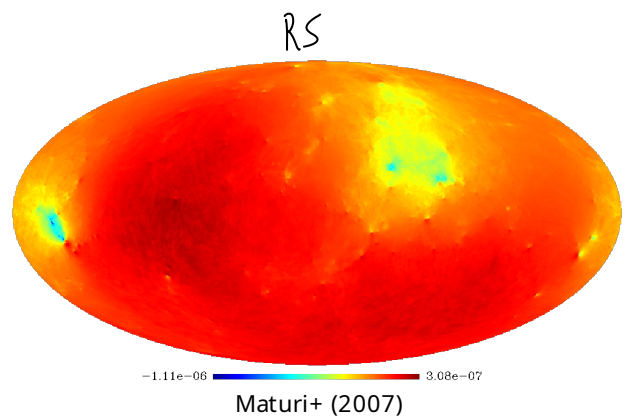
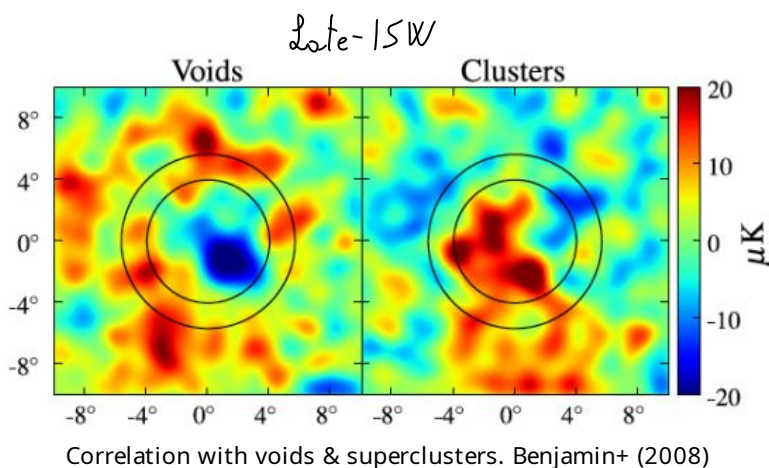
ISW $\rightarrow$ Early
Late

$z \approx z_{rec}$ $\Phi = \Phi_{DM} + \Phi_B + \Phi_R$ $\Phi_R \rightarrow 0$ suddenly
 $\Rightarrow \dot{\Phi} < 0 \Rightarrow \frac{\Delta T}{T} < 0$ $l \lesssim 1^\circ$ peak

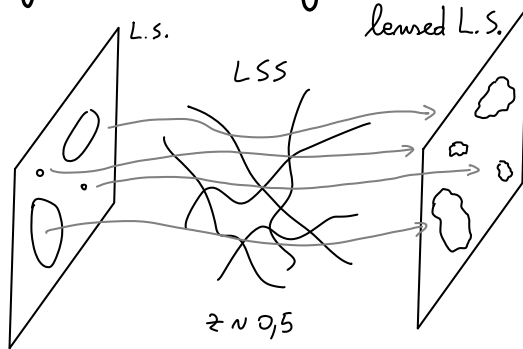
$z \ll z_{rec}$ $\Lambda > 0$ $\Phi \rightarrow \Phi + \Phi_\Lambda$
very large scales $\Rightarrow \Lambda$ affects low k

RS $\rightarrow$
Rees-Sciama

Non-linear scales $\Rightarrow$ collapse and cluster move
 $\Rightarrow$ very high k (small scales)



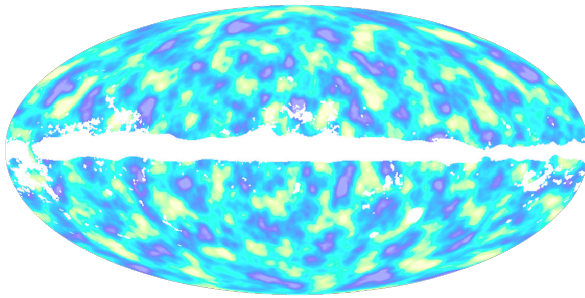
Gravitational lensing



- deflection angle $|\vec{\alpha}| \ll 1$
 $T(\vec{x}) \rightarrow \tilde{T}(\vec{x} + \vec{\alpha}) \simeq T(\vec{x}) + \vec{\nabla} T(\vec{x}) \vec{\alpha}$
- Change arrival position of photons
- Above certain angular scales is random
 $\Rightarrow$ "Smoothing": peaks are less sharp
- small effect, only $1 \sim 2\%$
- for $\Omega_m \uparrow$ and/or $\sigma_8 \uparrow$ stronger effect

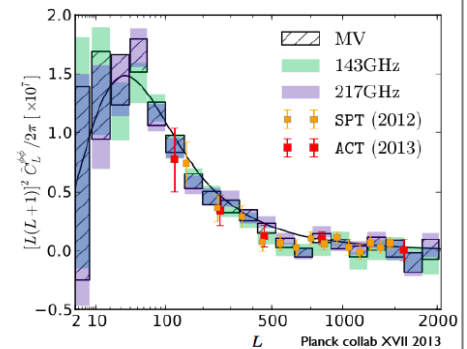
Also... CMB as a lensed source

weak lensing reconstruction: measure matter distribution
 power spectrum of reconstructed lensing potential



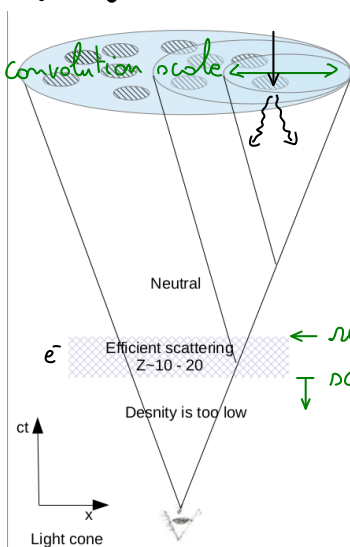
$$\tilde{\Theta}(\hat{n}) = \Theta(\hat{n} + \nabla\psi)$$

$$\tilde{C}_l^\Theta \approx \int dl C_{l-l'}^\psi C_{l'}^\Theta$$



Global Reionization $z \lesssim 10$ (?)

- Galaxy formation $\Rightarrow$ stars $\Rightarrow$ UV radiation $\Rightarrow$ ionization of Inter Galactic Medium (IGM)



Thomson scattering of CMB photons with free e^-

"Fog" convolves the peaks
 smaller scale fluctuations are diluted away
 amplitude of high l peaks is lowered

$\leftarrow$ reionization
 $\downarrow$ scattering is not relevant (low n_{e^-})

- The typical smoothing scale depends on z_{re} :
 lower $z_{re} \rightarrow$ larger convolution scale
 $\Rightarrow$ CMB measures constrain $z_{re} > 6$

▫ Sunjaev-Zel'dovich effect

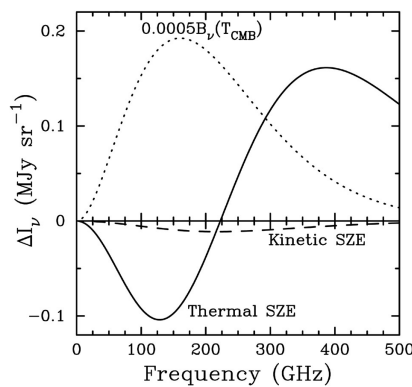
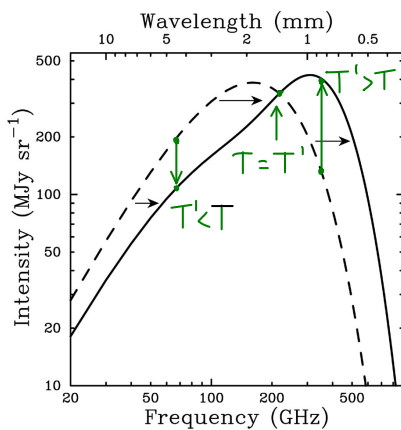
Within galaxy clusters : Intra Cluster Medium (ICM) is fully ionized $\Rightarrow$ free e^-

Kinematic SZ :

- Clusters have a peculiar motion
- scattering creates a doppler shift
- depends on n_e (small effect)

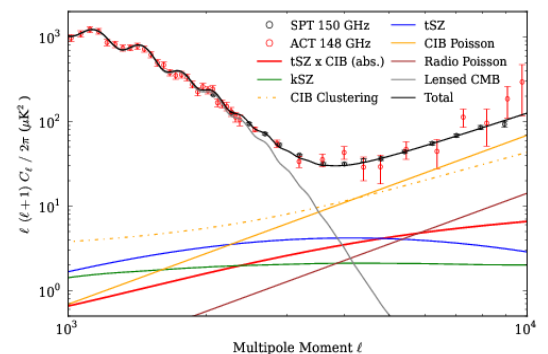
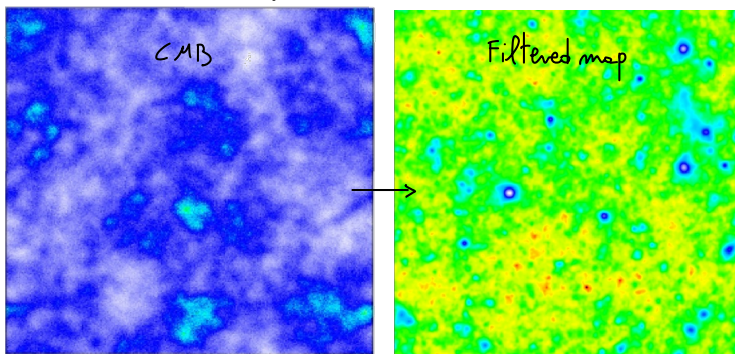
Thermal SZ :

- ICM has high T
 $\Rightarrow$ Inverse Compton scattering : e^- give energy to photons
- CMB black-body spectrum "shifted" toward higher ν
- Temperature change in photons is frequency dependent



Distortion of the CMB Planck spectrum

- Clusters Mass function
- Pressure profile of galaxy clusters, mass estimates



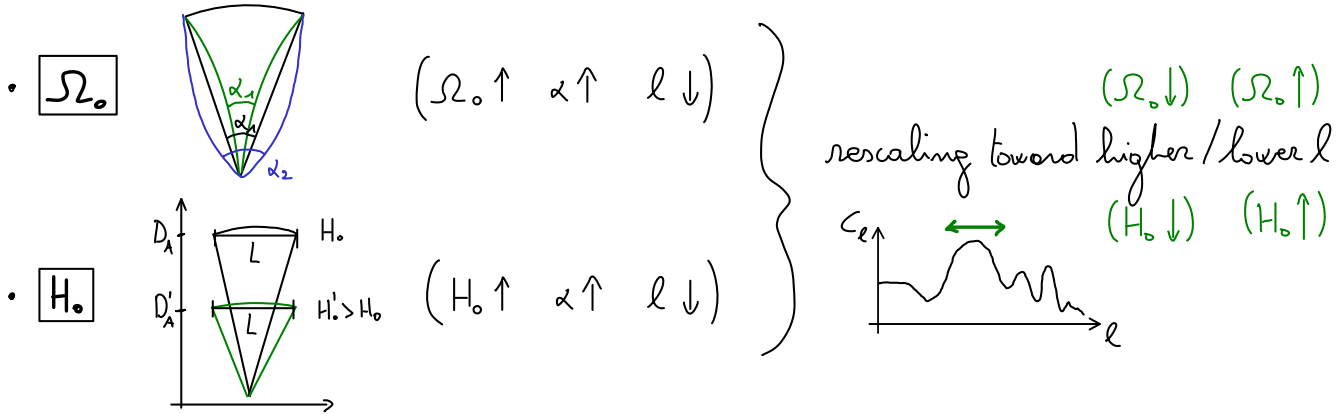
Main dependencies on the cosmological parameters

The features of the power spectrum depend on the cosmological parameters

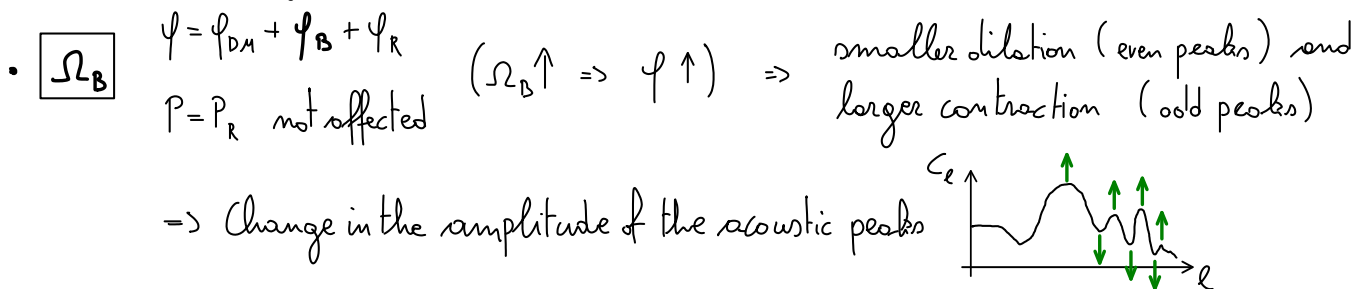
1) Because of the change in the angular diameter distance

the same physical scale L displays a different angle α

$$D_A = \frac{1}{H_0} \int E(z) dz$$



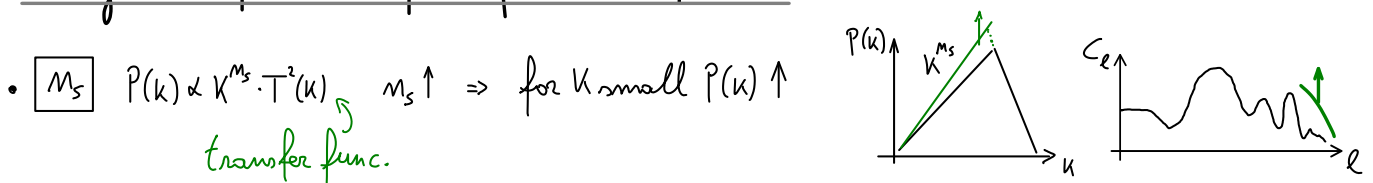
2) Change in the baryonic acoustic oscillations



3) Change in the dynamic of the universe



4) Change in the primordial power spectrum (inflation)

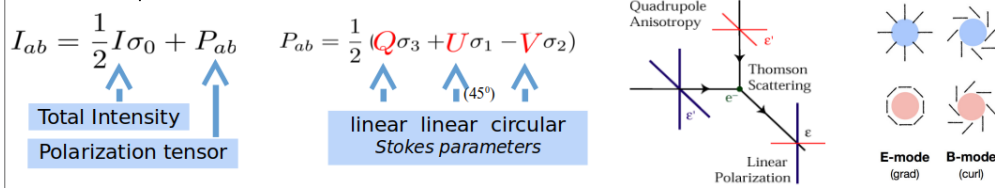


5) Gravitational lensing: $\Omega_M, \sigma_8 \uparrow \Rightarrow$ suppression of higher modes

6) Global reionization: $z_{re} \downarrow \Rightarrow$ suppression starts at lower l

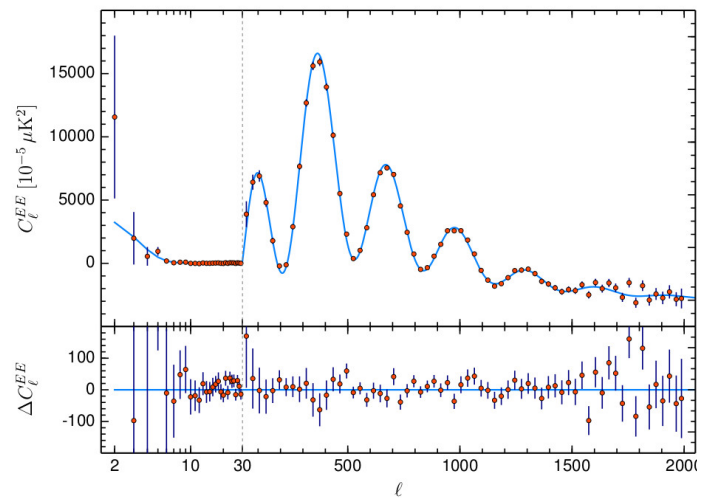
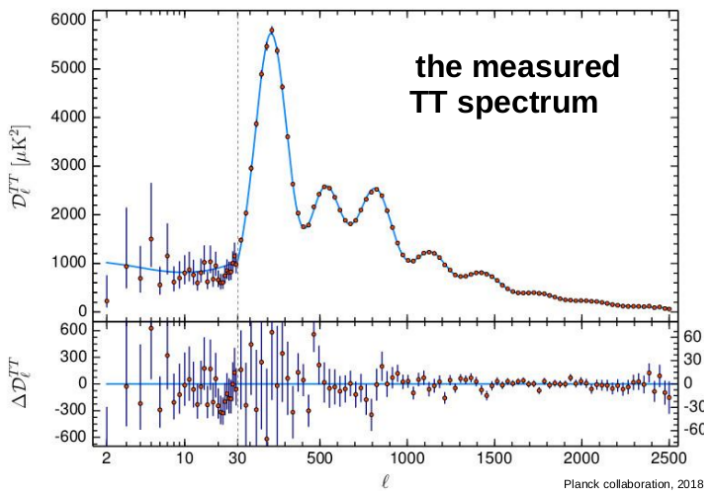
Polarization

Thomson scattering produces linearly polarized light when incoming radiation have a quadrupole momentum



Experiments

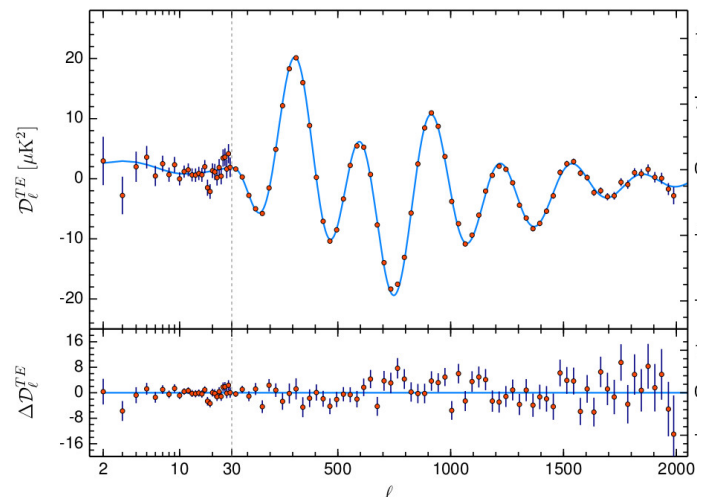
COBE : Planck spectrum, anisotropies 1992
 Boomerang : stratospheric balloon, curvature of the universe
 WMAP : higher resolution ($\sim 15'$)
 PLANCK : even higher resolution ($\sim 5'$) + Polarization
 SPT : ground based single dish ($\sim 1'$)
 ACT : ground based single dish ($\sim 1'$)



Polarization power spectra

$$C_l^{EE} = \frac{1}{2l+1} \sum_m |E_{lm}|^2 \quad C_l^{BB} = \frac{1}{2l+1} \sum_m |B_{lm}|^2$$

$$C_l^{TE} = \frac{1}{2l+1} \sum_m \Theta_{lm} E_{lm} \quad < \text{Cross correlation}$$





Gravitational lensing

Matteo Maturi

Heidelberg University

Cosmology WS 2025/26

Introduction

The key point

- Photons travel along null geodesics
- Matter fluctuations perturb the metric
- Photons follow **perturbed** null geodesics
- This is gravitational lensing

History: the beginnings

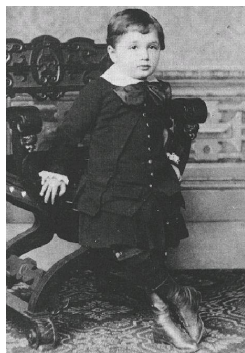
Photons as massive particles: John Mitchell (1784) $\Rightarrow \alpha = \frac{2GM}{c^2 r}$
Johann Von Soldner (1801)

Equivalence principle: Einstein (1915) $\Rightarrow \alpha = \frac{4GM}{c^2 r}$

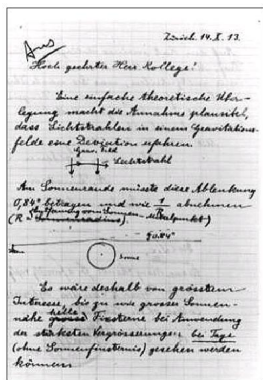
Experimentl proof: Dyson, Eddington and Davidson (1920) $\Rightarrow \alpha = 1''.75!$

"Dear Mother, – Good news today. H.A. Lorentz has wired me that the British expedition has actually proved the light deflection near the Sun."

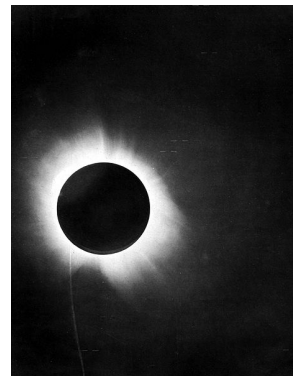
A. Einstein



A. Einstein



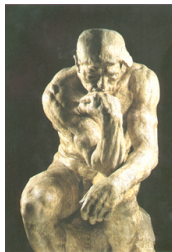
Einstein notes



Eddington Experiment

History: toward the modern times

Ideas about possible schenarios



"The thinker", Rodin

- Einstein rings (Chowolson, 1924)
- Multiple images
- Stellar microlensing (Mandl -Einstein, 1936)
- Galaxy (nebulae) lensing (Zwicky, 1937)
- M31 star by ML star (Liebes, 1964)
- Time delay (Refsdal, 1964)

The very first observations

Lensing by the Sun

(Dyson, Eddington, Davidson 1920)

QSO 0957+561

(Walsh, Carswell, Weymann, 1979)

- 2 images d=6" zs=1.41 identical spectra
- galaxy lens discovery zl=0.36
- VLBI → same core-jet structure

(Stockton, 1980)

Giant Arcs in A370 Cl2244

(Lynds & Peterosian 1986])

Rings

(Hewitt et al. 1988)

Quasar microlensing

(Irwin et al. 1989)

Weak lensing cluster

(Tyson, Valdes, Wenk 1990)

Weak lensing group of galaxies - LSS

(Brainderd, Blandford, Smail 1996)

Microensing star - star (ML bulge)

(Udalski et al. 1993)

Time delays QSO 095+561

(Kindic et al. 1997)

Gravitational lensing and optics

4-interval
$$ds^2 = c^2 dt^2 - d\vec{x}^2$$

$$ds^2 = \left(1 + \frac{2U}{c^2}\right) c^2 dt^2 - \left(1 - \frac{2U}{c^2}\right) d\vec{x}^2 = 0 \quad \left[v^2 = \frac{d\vec{x}^2}{dt^2} \right]$$

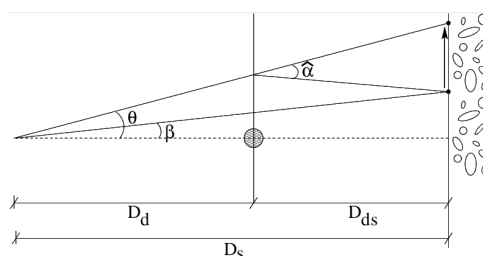
Diffraction index
$$n \equiv \frac{c}{v} = 1 + \frac{2}{c^2} |\Phi| \geq 1 \quad \Delta t = \int_o^s \frac{2}{c^3} |\Phi| dz$$

Deflection angle
$$\hat{\alpha} = - \int \vec{\nabla}_{\perp} n dz = \frac{2}{c^2} \int \vec{\nabla}_{\perp} \Phi dz$$

The lens equation

$$\vec{\beta}(\vec{\theta}) = \vec{\theta} - \frac{D_{ds}}{D_s} \vec{\alpha}(\theta) = \vec{\theta} - \vec{\tilde{\alpha}}(\theta)$$

Assuming: $\theta, \beta, \alpha \ll 1$



- **Thin lens approximation:** lens size much smaller than distances
- It relates the lens, the source and its image positions on the sky
- Describes how a deflection field acts on a background source.
- $\vec{\tilde{\alpha}}$ is the **reduced deflection angle**

The lensing potential: deflection

- The *lensing potential* is defined as:

$$\psi(\vec{\theta}) \equiv \frac{2}{c^2} \frac{D_{ds}}{D_d D_s} \int \Phi(D_d \vec{\theta}, z) dz$$

gravitational potential integrated along the line of sight (thin lens!)

- The *reduced deflection angle* $\vec{\alpha}$ is the gradient of the lens potential

$$\vec{\nabla}_{\theta} \psi(\vec{\theta}) = \frac{2}{c^2} \frac{D_{ds}}{D_d D_s} \int \vec{\nabla}_{\theta} \Phi dz = \frac{D_{ds}}{D_s} \frac{2}{c^2} \int \vec{\nabla}_{\perp} \Phi dz = \frac{D_{ds}}{D_s} \vec{\alpha}(\vec{\theta})$$

$$\vec{\nabla}_{\theta} \psi(\vec{\theta}) = \vec{\tilde{\alpha}}(\vec{\theta})$$

- The *convergence* κ is the laplacian of the lens potential

$$\nabla_{\theta}^2 \psi(\vec{\theta}) = \frac{2}{c^2} \frac{D_d D_{ds}}{D_s} \int \nabla_{\theta}^2 \Phi dz = \frac{2}{c^2} \frac{D_d D_{ds}}{D_s} 4\pi G \int \nabla_{\theta}^2 \varrho dz = \frac{2}{c^2} \frac{D_d D_{ds}}{D_s} 4\pi G \Sigma(\vec{\theta})$$

$$\nabla_{\theta}^2 \psi(\vec{\theta}) = 2\kappa(\vec{\theta}) \quad \kappa(\vec{\theta}) \equiv \Sigma(\vec{\theta})/\Sigma_{cr} \quad \Sigma_{cr} \equiv \frac{c^2}{4\pi G} \frac{D_s}{D_d D_{ds}}$$

κ quantifies the surface mass density

Here, we used the Poisson eq. $\nabla^2 \Phi = 4\pi G \varrho$ and defined the critical density Σ_{cr}

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Lens mapping: shear and convergence

- The lens equation can be **linearized** for small α

$$\vec{\beta}(\vec{\theta}) = \vec{\theta} - \frac{D_{ds}}{D_s} \vec{\alpha}(\theta) = \vec{\theta} - \vec{\tilde{\alpha}}(\theta) \quad \rightarrow \quad \vec{\beta}(\vec{\theta}) \sim \vec{\beta}_0 + A(\vec{\theta}_0)(\vec{\theta} - \vec{\theta}_0)$$

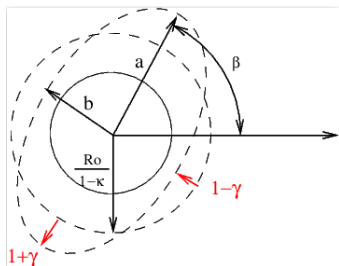
- The image *transformation* of the source is described by the *Jacobian matrix* of the lens equation

$$A \equiv \frac{\partial \vec{\beta}}{\partial \vec{\theta}} = \left(\delta_{ij} - \frac{\partial^2 \psi(\vec{\theta})}{\partial \theta_i \partial \theta_j} \right) = \begin{pmatrix} 1 - \kappa - \gamma_1 & -\gamma_2 \\ -\gamma_2 & 1 - \kappa + \gamma_1 \end{pmatrix} = (1 - \kappa) \begin{pmatrix} 1 - g_1 & g_2 \\ g_2 & 1 + g_1 \end{pmatrix}$$

- isotropic magnification (convergence) and anisotropic distortion (shear) terms

$$A = (1 - \kappa) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix}$$

- Here, we introduced the **shear** $\gamma = \gamma_1 + i\gamma_2 = |\gamma|e^{2i\phi}$



$$\gamma_1 = \frac{1}{2}(\psi_{,11} - \psi_{,22}) \quad \gamma_2 = \frac{1}{2}(\psi_{,12} + \psi_{,21})$$

$$g_i = \frac{\gamma_i}{1 - k} \quad \text{reduced shear}$$

Image magnification

- **Magnification** (defined as the ratio between solid angles)

$$\begin{aligned} a &= r(1 - \kappa + \gamma)^{-1} \\ b &= r(1 - \kappa - \gamma)^{-1} \end{aligned}$$

$$\begin{aligned} \Omega_I &= \pi ab = \pi r^2 / \det A \\ \Omega_0 &= \pi r^2 = \pi \end{aligned} \quad \mu = \frac{\Omega_I}{\Omega_0} = \frac{1}{\det A} = \frac{1}{(1 - \kappa)^2 - \gamma^2}$$

- **Magnification tensor:**

$$M = A^{-1}$$

Its eigenvalues measure the amplification along the tangential and radial directions:

$$\mu_t = \frac{1}{\lambda_t} = \frac{1}{1 - \kappa - \gamma} \quad \mu_r = \frac{1}{\lambda_r} = \frac{1}{1 - \kappa + \gamma}$$

- **Strong lensing and the critical curves:**

magnification diverges along the curves where $\lambda_t = 0$ $\lambda_r = 0$

The mass sheet degeneracy

- **Homogeneous mass distributions** do not introduce any image distortion
- Clear **invariance transformation** of the lens mapping:

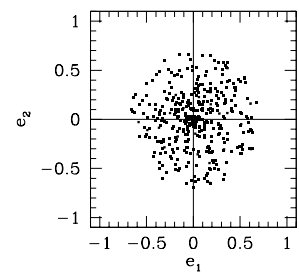
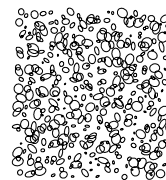
$$A \rightarrow \lambda A = \lambda(1 - \kappa) \begin{pmatrix} 1 - g_1 & g_2 \\ g_2 & 1 + g_1 \end{pmatrix} = (1 - \kappa') \begin{pmatrix} 1 - g_1 & g_2 \\ g_2 & 1 + g_1 \end{pmatrix}$$

- The new convergence is $\kappa' = 1 - \lambda(1 - \kappa)$
- The new shear is $\gamma' = \lambda\gamma$
- **The new reduced shear (what we observe) is unchanged $g' = g$**
- One way to break this degeneracy is to combine shear and magnification estimates

Looking at the sky: galaxies as background sources

Distant galaxies
Typical size
Typical density

$$\begin{aligned} (z &\sim 1 - 2) \\ r &\sim 1'' - 10'' \\ n &\sim 10 - 100 \frac{\text{gal}}{\text{arcmin}^2} \end{aligned}$$

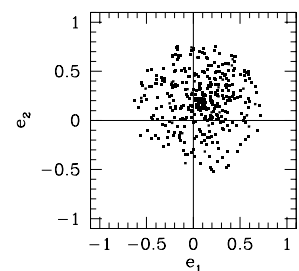


Distorted ellipticity

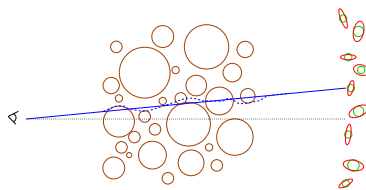
$$\epsilon = \frac{\epsilon^{(s)} + g}{1 + g^* \epsilon^{(s)}}$$

Hypothesis:

$$\langle \epsilon^{(s)} \rangle = 0 \quad \rightarrow \quad \langle \epsilon \rangle = g$$



Gravitational lensing by LSS



Refregier 2003, ARA&A,41,645R

- Linear structures
- Small density contrast: $\delta \equiv \frac{\rho - \rho_b}{\rho_b} \ll 1$
- Characterized by large scales
- Fill the whole line of sight: sum over all contributions

- So far we concentrated our attention on single lenses
- But the universe is filled with distributed matter
- Its distribution depends on cosmology $\rightarrow$ let's study it!
- The effective convergence is given by the sum of lens planes

$$\kappa_{\text{eff}}(\vec{\theta}, w) = \frac{3H_0^2 \Omega_m}{2c^2} \int_0^w dw' \frac{f_K(w - w') f_K(w')}{f_K(w)} \frac{\delta[f_K(w') \vec{\theta}, w']}{a(w')} .$$

Reminder:

$f_K(w)$: The comoving angular-diameter distance

$\delta \equiv (\rho - \bar{\rho})/\bar{\rho}$: density contrast

$a(w)$: scale factor of the universe at distance w .

$2\kappa(\vec{\theta}) = \nabla_{\vec{\theta}}^2 \psi(\vec{\theta})$: convergence

$\psi(\vec{\theta}) \equiv \frac{2}{c^2} \frac{D_{ds}}{D_d D_s} \int \Phi(D_d \vec{\theta}, z) dz$ and $\nabla^2 \Phi = 4\pi G \rho$: lensing potential

Navigation icons: back, forward, search, etc.

Cosmic shear power spectrum

- **Sources are distributed in redshift:** weight over the source–distance distribution $G(w)$,

$$\bar{\kappa}_{\text{eff}}(\vec{\theta}) = \int_0^{w_H} dw G(w) \kappa_{\text{eff}}(\vec{\theta}, w)$$

- **The power spectrum** of the effective convergence $P_{\kappa}(k)$ is related to the matter power spectrum $P_{\delta}(k)$ by the **Limber's equation**,

$$P_{\kappa}(k) = \frac{9H_0^2 \Omega_m^2}{4c^2} \int_0^{w_H} dw \frac{\bar{W}^2(w)}{a^2(w)} P_{\delta}\left(\frac{k}{f_K(w)}, w\right)$$

the **weight function** $\bar{W}(w)$ is given by

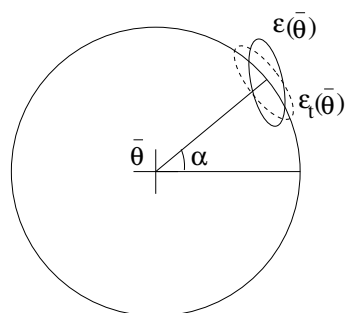
$$\bar{W}(w) = \int_w^{w_H} dw' G(w') \frac{f_K(w' - w)}{f_K(w')}$$



The aperture mass

$$M_{\text{ap}}(\vec{\theta}) = \int d^2\theta' \gamma_t(\vec{\theta}') Q(\vec{\theta} - \vec{\theta}')$$

- Q is defined to be compensated
- Info on surface mass density (κ)
- Detection of clusters (candidates)



Lens mapping

- Assumption: sources are much smaller than the scale on which lensing changes

$\Rightarrow$ Linearize $\bar{\beta} = \bar{\theta} - \bar{\alpha}(\bar{\theta}) = \bar{\theta} - \bar{\nabla}\psi(\bar{\theta})$ at 1st order $\delta\bar{\beta} \approx A\delta\bar{\theta}$
 $\delta\bar{\theta} \sim A^{-1}\delta\bar{\beta}$ (where A is invertible)

- Jacobian matrix of the lens mapping

$$A_{ij} = \frac{\delta\bar{\beta}_i}{\delta\bar{\theta}_j} = \frac{\delta\bar{\theta}_i}{\delta\bar{\theta}_j} - \frac{\delta}{\delta\bar{\theta}_j} \frac{\delta}{\delta\bar{\theta}_i} \psi(\bar{\theta}) = \delta_{ij} - \gamma_{ij}(\bar{\theta}) \quad \gamma_{ij} = \frac{\delta^2\psi(\bar{\theta})}{\delta\bar{\theta}_i\delta\bar{\theta}_j} \quad \text{if } \gamma_{ij} = 0 \Rightarrow A = I \text{ identity} \\ \Rightarrow \text{no lensing}$$

- Decompose in isotropic and Anisotropic (trace free) components

$$(1) \quad \text{Tr}(A) = 1 - \gamma_{11} + 1 - \gamma_{22} = 2 - \gamma_{11} - \gamma_{22} = 2 - \bar{\nabla}^2\psi = 2(1 - \kappa) \quad \kappa \equiv \frac{1}{2}\bar{\nabla}^2\psi$$

$$(2) \quad \mathcal{T} \equiv -\left(A - \frac{1}{2}\text{Tr}(A) \cdot I_2\right) \quad \mathcal{T}_{11} = -\left[\cancel{1} - \gamma_{11} - \frac{1}{2}(\cancel{2} - \gamma_{11} - \gamma_{22})\right] = \frac{1}{2}(\gamma_{11} - \gamma_{22}) \equiv \gamma_1$$

$$\mathcal{T}_{22} = -\left[\cancel{1} - \gamma_{22} - \frac{1}{2}(\cancel{2} - \gamma_{11} - \gamma_{22})\right] = \frac{1}{2}(\gamma_{22} - \gamma_{11}) = -\gamma_1$$

$$\mathcal{T}_{12} = \gamma_{12} = \gamma_{21} = \frac{1}{2}(\gamma_{12} + \gamma_{21}) \equiv \gamma_2 \quad \text{spin 2 (pseudovector)}$$

$$\Rightarrow \kappa = \text{convergence}, \quad \gamma \equiv \gamma_1 + i\gamma_2 = |\gamma| e^{i2\phi} \quad \text{Complex shear}$$

$$\Rightarrow A = (1 - \kappa)I - \mathcal{T} = (1 - \kappa) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -\gamma_1 & -\gamma_2 \\ -\gamma_2 & \gamma_1 \end{pmatrix} = \begin{pmatrix} 1 - \kappa - \gamma_1 & -\gamma_2 \\ -\gamma_2 & 1 - \kappa + \gamma_1 \end{pmatrix}$$

$\circ \rightarrow \bigcirc \quad \quad \quad \circ \rightarrow \bigcirc$

- in weak lensing regime: $\det(A) = (1 - \kappa)^2 - \gamma^2 \approx 1$

- But what we see is $\bar{\theta}$ and not $\bar{\beta}$!

$\Rightarrow$ inverse lensing $\delta\bar{\theta} = A^{-1}\delta\bar{\beta}$ which is what we "see": $A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} 1 - \kappa + \gamma_1 & \gamma_2 \\ \gamma_2 & 1 - \kappa - \gamma_1 \end{pmatrix}$
 $\uparrow$ observable $\uparrow$ invertible if $\det(A) \neq 0$ $= \mu$ magnification

Cosmic shear

- $\bar{\kappa}'(\bar{\theta}, w) = f_k(w) \bar{\theta}$ Unperturbed light bundle $\kappa = \text{curvature}$
 $\swarrow$
 $\begin{cases} L & \text{comoving distance} \\ & \text{angular diameter distance} \end{cases}$
- $\bar{\kappa}(\bar{\theta}, w) = f_k(w) (\bar{\theta} - \bar{\alpha}_{\text{eff}})$ Perturbed light bundle, $\bar{\alpha}_{\text{eff}}$ effective deflection angle ^(reduced)
- $\bar{\alpha}_{\text{eff}}(\bar{\theta}, w) = \frac{f_k(w) \bar{\theta} - \bar{\kappa}(\bar{\theta}, w)}{f_k(w)} = \bar{\nabla}_{\bar{\theta}} \psi = \frac{2}{c^2} \int_0^{z(w)} \frac{D_{ds}}{D_s} \bar{\nabla}_{\perp} \psi dz' = \frac{2}{c^2} \int_0^w \frac{f_k(w-w')}{f_k(w)} \bar{\nabla}_{\perp} \psi [f_k(w') \bar{\theta}, w'] dw'$
- $\kappa_{\text{eff}}(\bar{\theta}, w) = \frac{1}{2} \bar{\nabla}_{\bar{\theta}} \bar{\alpha}_{\text{eff}}(\bar{\theta}, w) = \frac{1}{c^2} \int_0^w \frac{f_k(w-w') f_k(w')}{f_k(w)} \bar{\nabla}_{\perp}^2 \psi [f_k(w') \bar{\theta}, w'] dw'$ Effective convergence
 $\uparrow$ source plane $\bar{\nabla}_{\bar{\theta}} = f_k(w') \bar{\nabla}_{\perp}$ 3-D density contrast
 $\bar{\nabla}_{\perp}^2 \psi = \frac{3 H_0^2 \Omega_0}{2 a} \delta$
 $\delta \equiv \frac{\rho - \langle \rho \rangle}{\langle \rho \rangle}$

With distributed sources

$$\begin{aligned}
 \bar{\kappa}_{\text{eff}}(\bar{\theta}) &= \int_0^{w_H} dw G(w) \kappa_{\text{eff}}(\bar{\theta}, w) & G: \text{Sources redshift distribution} \\
 &= \frac{3 H_0^2 \Omega_0}{2 c^2} \int_0^{w_H} dw G(w) \int_0^w \frac{f_k(w-w') f_k(w')}{a(w') f_k(w)} \delta[f_k(w') \bar{\theta}, w'] dw' \\
 &= \frac{3 H_0^2 \Omega_0}{2 c^2} \int_0^{w_H} \left[\int_0^w dw G(w) \frac{f_k(w-w')}{f_k(w)} \right] f_k(w') \frac{\delta[f_k(w') \bar{\theta}, w']}{a(w')} dw' \\
 &= \frac{3 H_0^2 \Omega_0}{2 c^2} \int_0^{w_H} W(w') f_k(w') \frac{\delta[f_k(w') \bar{\theta}, w']}{a(w')} dw' & W(w') \equiv \int_{w'}^{w_H} G(w) \frac{f_k(w-w')}{f_k(w)} dw
 \end{aligned}$$

$w = \text{source plane}$ $w' = \text{lens plane}$ $w' < w$

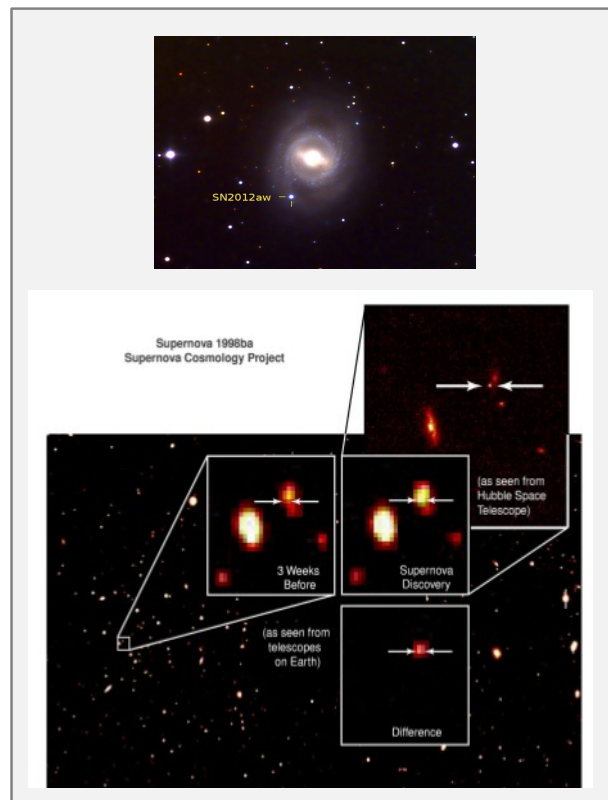
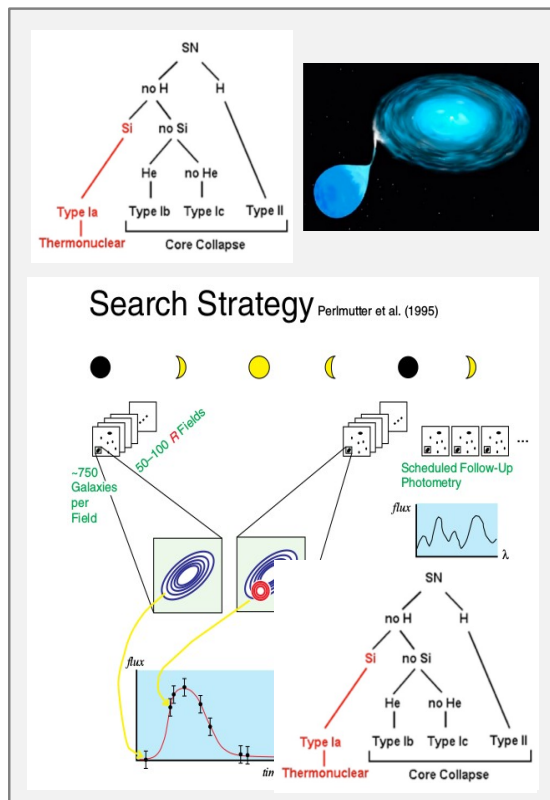
Projection: Limberg's eq. $\int 3D \rightarrow \int 2D$

$$P_k(\ell) = \frac{9 H_0^4 \Omega_0^2}{4 c^4} \int_0^{w_H} \frac{W^2(w')}{a^2(w')} f_k^2(w') P_{\delta}\left(\frac{\ell}{f_k(w')}, w'\right) dw' \quad P_k(\ell) = P_{\delta}(\ell) \quad \text{because they are both } 2^{\text{nd}} \text{ order derivatives of } \psi$$

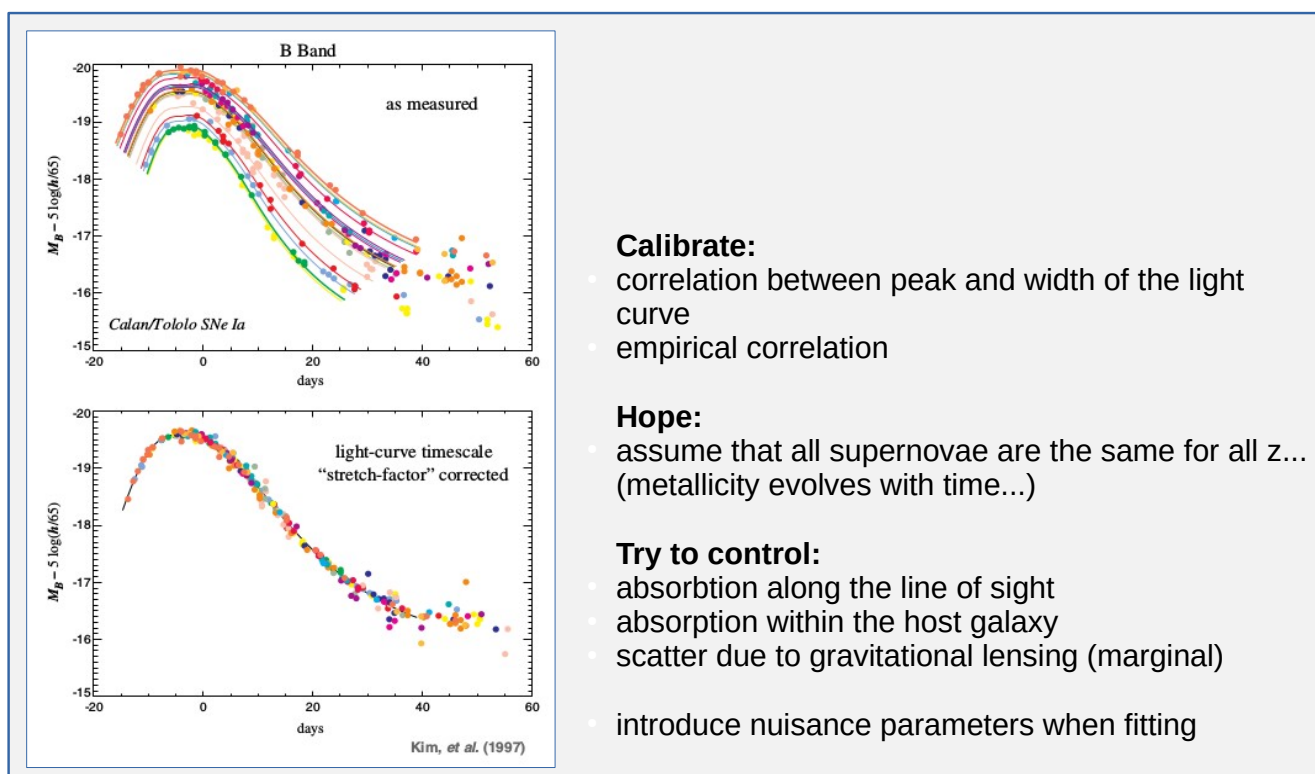
- Aperture mass dispersion: $\langle M_{\text{ap}}^2 \rangle(\theta) = 2\pi \int_0^{\infty} \ell d\ell P_k(\ell) J^2(\ell \theta)$ $J(x) = \frac{12}{\pi x^2} J_4(x)$
 Another related statistic $\uparrow$ 4° Bessel function

Luminosity distance

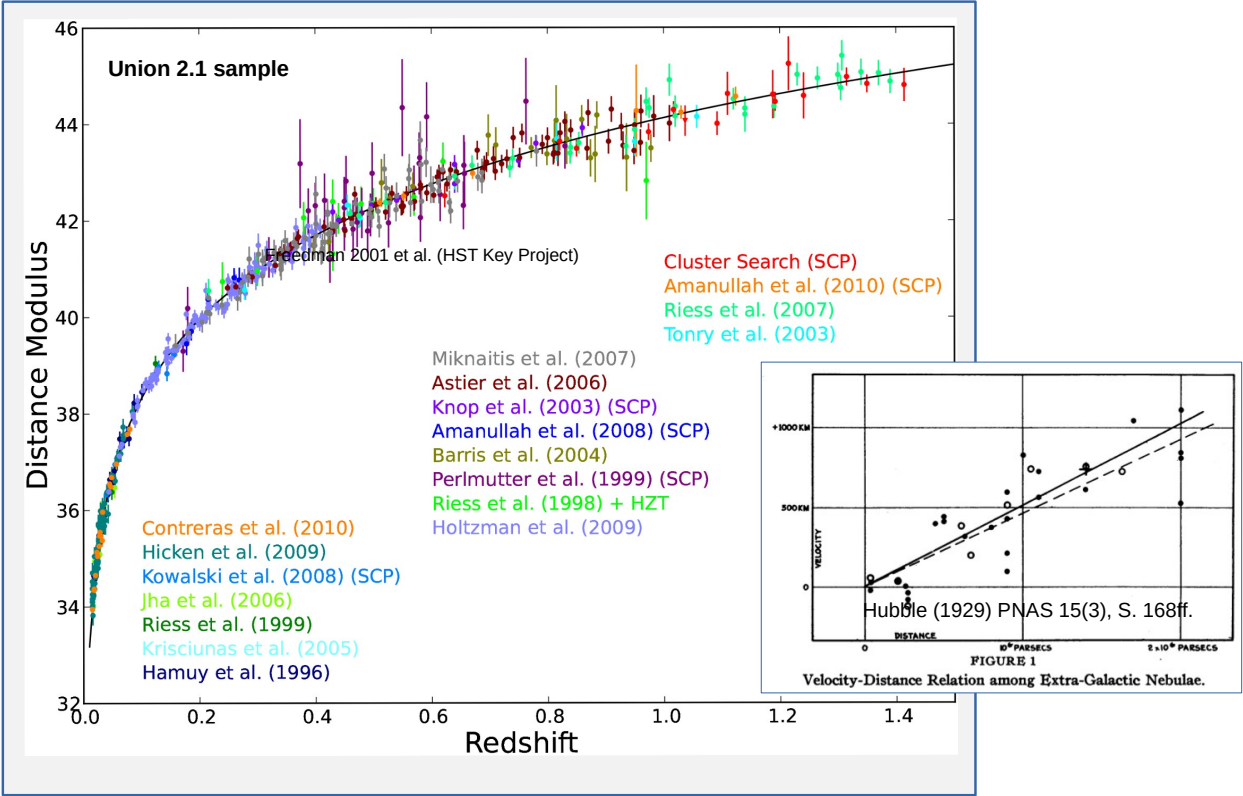
Supernovae Ia



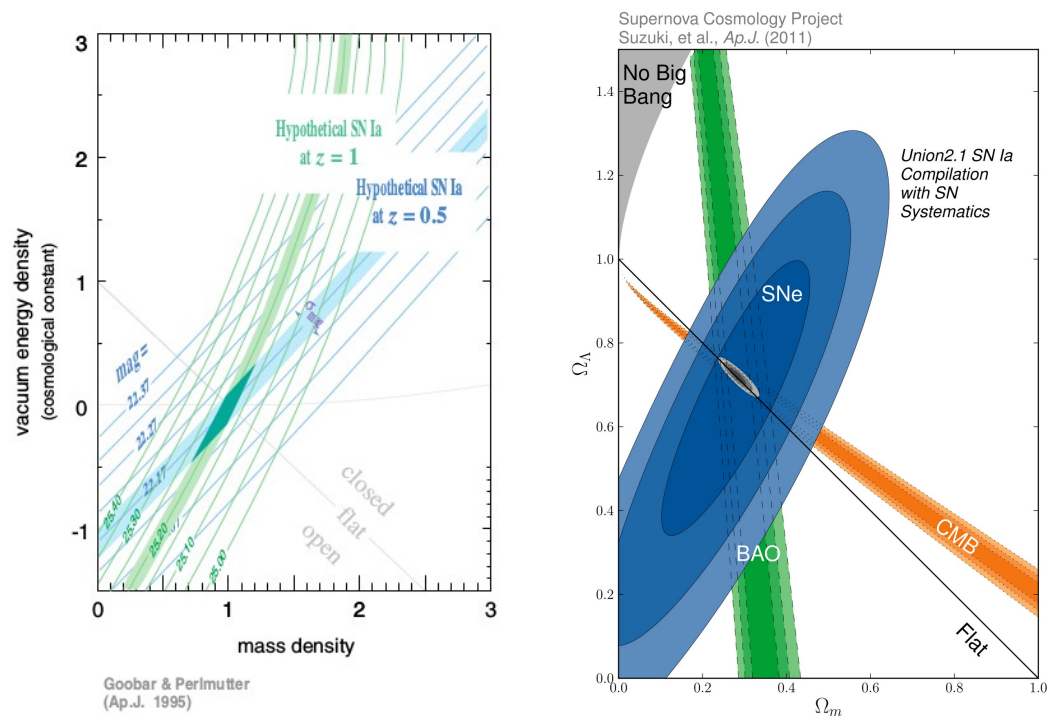
Standardizable not standard candles!



Super Novae



Understanding the “banana plots”



Model free expansion rate

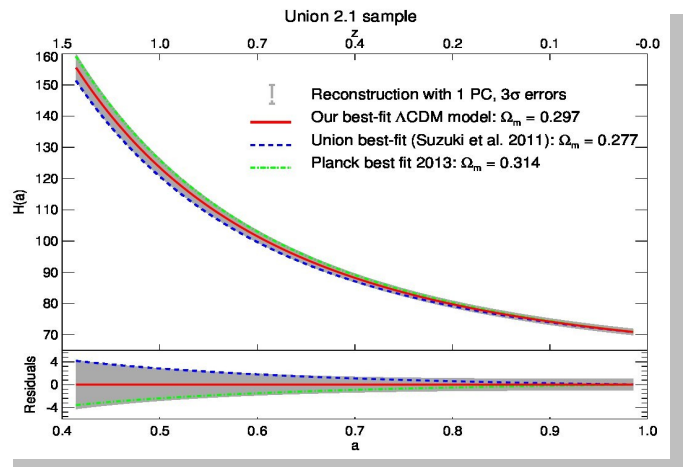
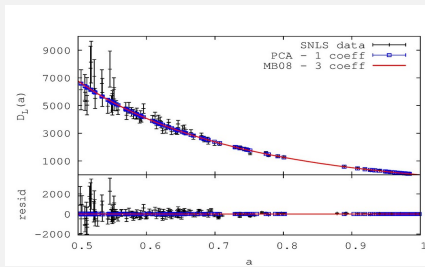
$$D_L(a) = \frac{c}{H_0} \frac{1}{a} \int_a^1 \frac{dx}{x^2 E(x)} \equiv \frac{c}{H_0} \frac{1}{a} \int_a^1 \frac{dx}{x^2} e(x),$$

Luminosity function
(Volterra integral of the second kind)

$$e(a) = -a^3 D'_L(a) + a \int_1^a \frac{dx}{x^2} e(x). \quad e(a) = \sum_{i=0}^{\infty} a^i f_i(a)$$

Solved in terms of a Neumann series

$$D_L(a) = \sum_{j=0}^J c_j p_j(a). \quad D'_L(a) = \sum_{j=0}^L c_j w'_j(a).$$





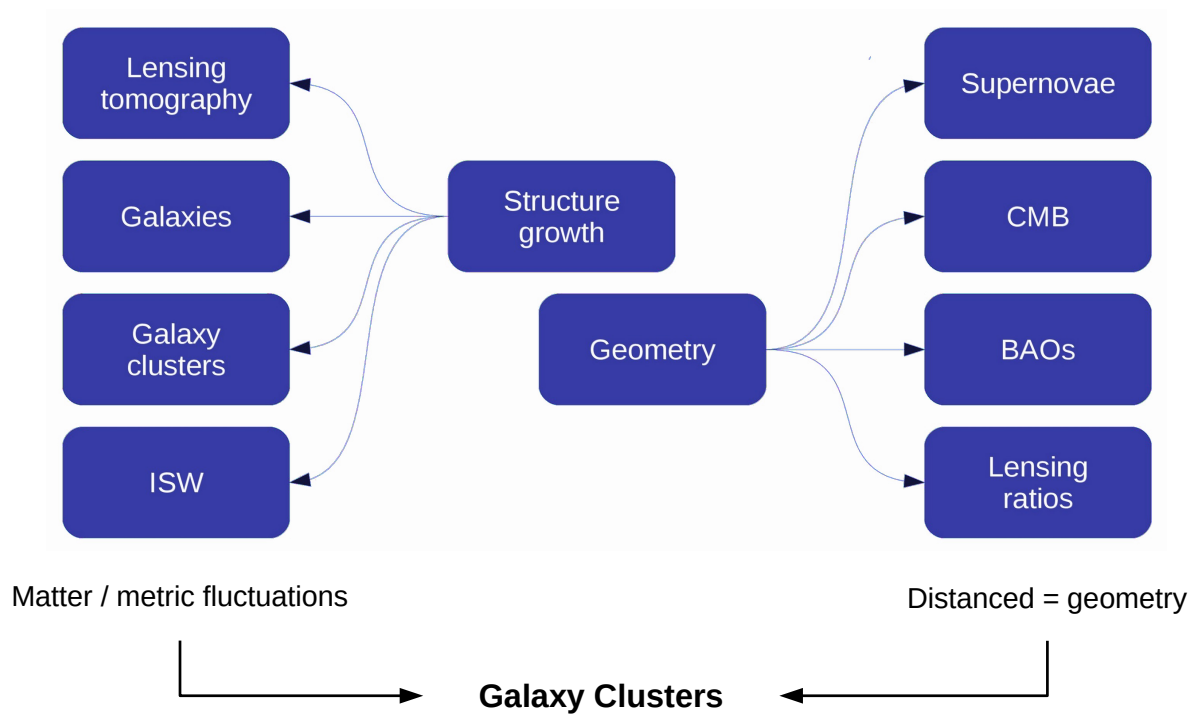
Cosmology with optical galaxy clusters

Matteo Maturi
Heidelberg University

Cosmology 2025/26

The GAME

Cosmology, how to





Bertone and Tail (2018)

Kill the dragons... Scalar-tensor theory: Horndeski

$$S = \int d^4x \sqrt{-g} \sum_{i=2}^5 \mathcal{L}_i + S_m$$

Horndeski (1974)

$$X = -g_{\mu\nu} \phi^{,\mu} \phi^{,\nu} / 2$$

← canonical kinetic term

$$\mathcal{L}_2 = K(\phi, X),$$

← non-canonical kinetic term

$$\mathcal{L}_3 = -G_3(\phi, X) \square \phi,$$

← non-canonical kinetic term G_3

$$\mathcal{L}_4 = G_4(\phi, X) R + G_{4,X} \left[(\square \phi)^2 - (\nabla_\mu \nabla_\nu \phi)^2 \right],$$

← couplings ↴

$$\mathcal{L}_5 = G_5(\phi, X) G_{\mu\nu} \nabla^\mu \nabla^\nu \phi - \frac{G_{5,X}}{6} \left[(\square \phi)^3 - 3(\square \phi) (\nabla_\mu \nabla_\nu \phi)^2 + 2(\nabla_\mu \nabla_\nu \phi)^3 \right]$$

The most general scalar-tensor theory giving second order equations of motion

Propagation of tensor modes (GW): $c_T = 1$ for $G_4 = G_5 = 0$

Why complicating gravity?

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G \langle T_{\mu\nu} \rangle_{\text{bone}}$$

$G_{\mu\nu} + \Lambda g_{\mu\nu}$: classical geometrical terms

$\langle T_{\mu\nu} \rangle_{\text{bone}}$: QFT expectation value of vacuum and all species

In flat space-time $\langle 0|T_{00}|0\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2} \hbar \omega_k \rightarrow \infty$ vacuum expectation value of hamiltonian

In curved space-time it is worst $\langle T_{\mu\nu} \rangle_{\text{bone}} = \langle T_{\mu\nu} \rangle_{\text{finite}} + \langle T_{\mu\nu} \rangle_{\text{div}}$ contains divergent terms arising from curvature

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + \alpha_2 H_{\mu\nu}^{(2)} + \dots = 8\pi G \langle T_{\mu\nu} \rangle_{\text{finite}} \quad \langle T_{\mu\nu} \rangle_{\text{bone}} = (\text{finite state-dependent terms}) + \underbrace{\frac{\Lambda_0^4}{16\pi^2} g_{\mu\nu} + \frac{\Lambda_0^2}{16\pi^2} G_{\mu\nu} + \ln(\Lambda_{UV}) H_{\mu\nu}^{(0)} + \dots}_{\text{particles (excited states)}} \quad \begin{matrix} A_0 & A_1 & A_2 \end{matrix}$$

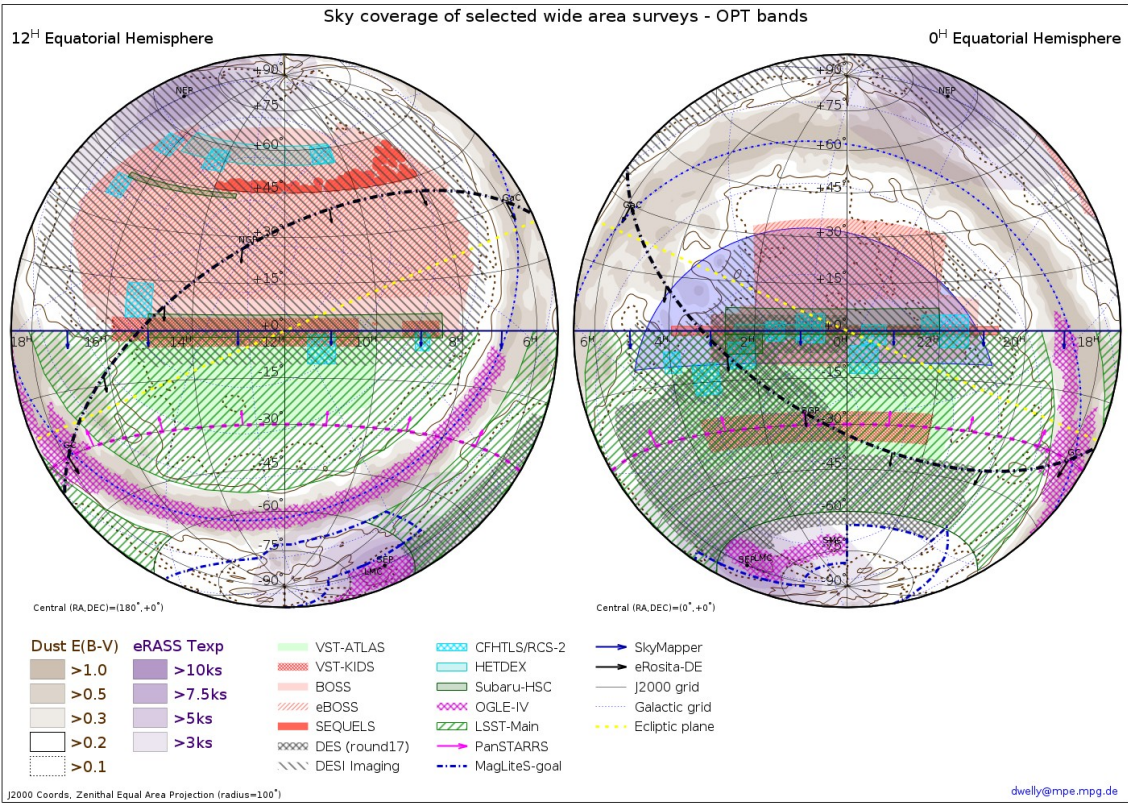
S(tensor field(s), baryons, radiation, scalar field_{dm}, scalar field_Λ, inflaton, interactions)

Optical / NIR surveys

Survey	Filters & depths	Area (deg ²)	Seeing (")	n _{eff} (gal/arcmin ²)
Euclid	VIS ~24.5; Y ~24.0; J ~24.0; H ~24.0	15 000	0.18 (VIS)	30
Roman HLIS	R062 ~26.9; Z087 ~26.9; Y106 ~26.9; J129 ~26.9; H158 ~26.9; F184 ~26.9	2 000	0.12	45
HSC Wide	g ~26.5; r ~26.1; i ~25.9; z ~25.1; y ~24.4	1 400	0.6	25–30
DES	g ~24.7; r ~24.4; i ~23.8; z ~23.1; Y ~21.7	5 000	1.0	6–7
KIDS + VIKING	u ~24.2; g ~25.1; r ~24.9; i ~24.2; ZYJHK ~22–23	1 347	0.7	8
LSST (10-yr)	u ~26.3; g ~27.5; r ~27.7; i ~27.0; z ~26.3; y ~24.9	18 000	0.7	26–30
J-PAS (almost spectroscopy)	54 NB filters to r ~23.0	8 000 (goal)	1.0	~3–4

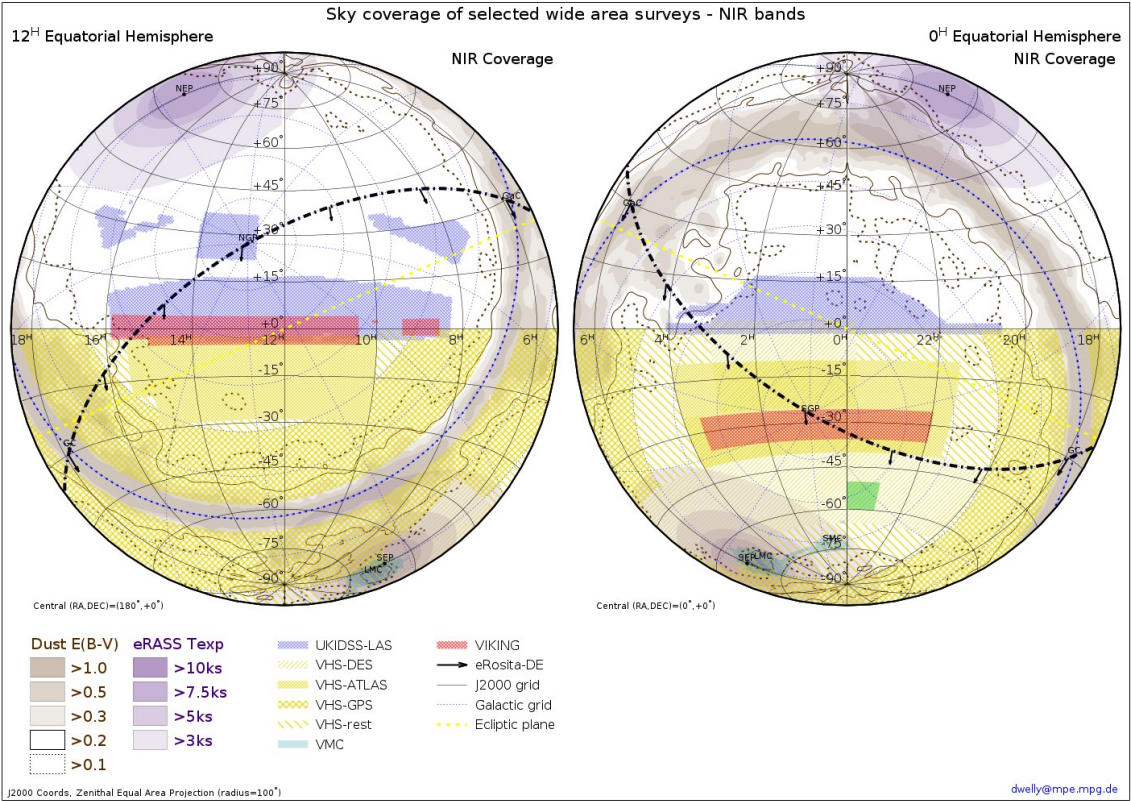
↓

Footprint
optical



Footprint
NIR

Euclid!!!

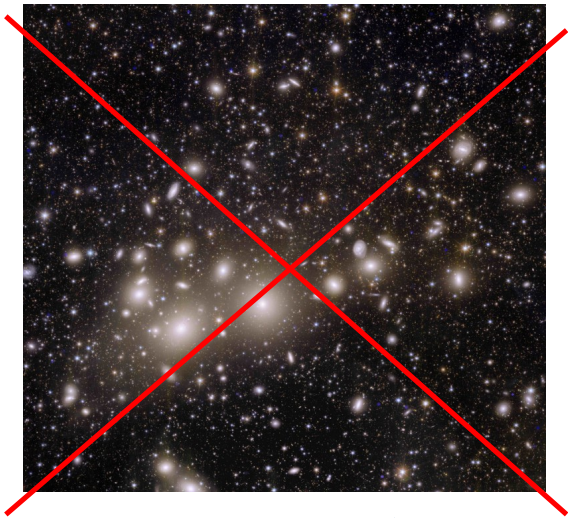


Cluster samples

Optical detections



Optical detections



Galaxies

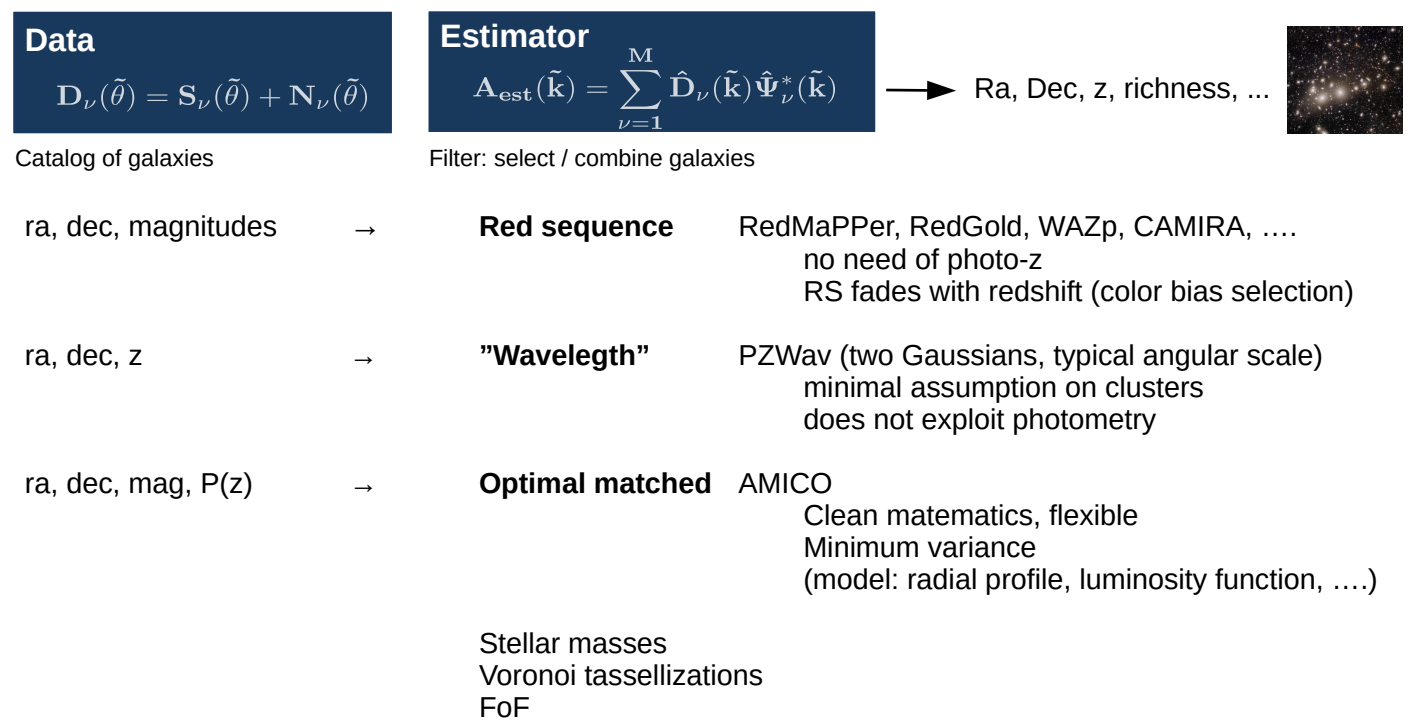


Stars

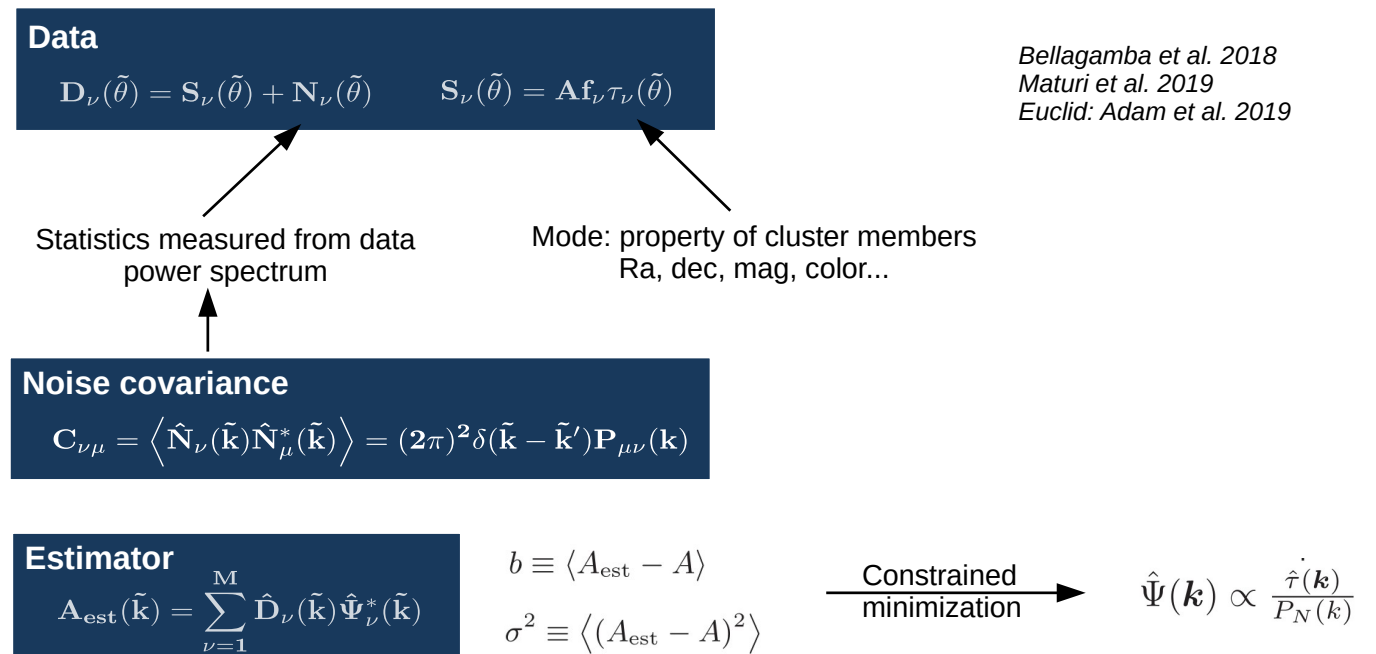
3706	7463.61	157.232	2	1.25	-0.018673	0.0329224	6.87081e-06	10.2722	-0.00933983	0.9
4079	785.072	160.885	2	1.25	-0.00677133	-0.0191166	1.33513e-05	10.2722	-0.00338601	-0.9
4126	4863.13	204.884	2	1.25	-0.00677133	-0.0191166	1.86214e-05	10.2722	-0.0228858	0.9
4248	2885.11	215.226	2	1.25	-0.00677133	-0.0191166	1.42859e-05	10.2722	-0.00403329	0.9
4792	6984.93	237.745	2	1.25	-0.00677133	-0.0191166	1.19831e-06	10.2722	-0.0199633	0.9
5372	2113.37	282.304	2	1.25	-0.00677133	-0.0191166	1.15905e-06	10.2722	-0.0104369	0.9
5475	7424.76	301.021	2	1.25	-0.0436674	0.0340955	9.44324e-06	10.2722	-0.0218505	0.9
5956	9548.46	321.836	2	1.25	-0.0436195	-0.0446261	4.49729e-06	10.2722	-0.021831	-0.9

vectors

Optical detections in general

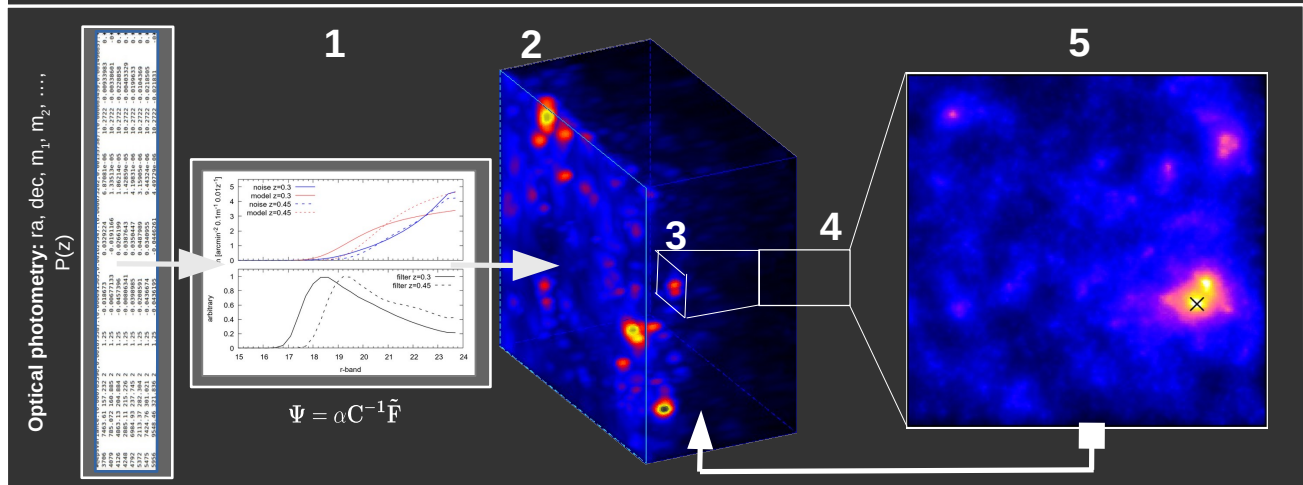


AMICO: Adaptive Matched Identifier of Clustered Objects



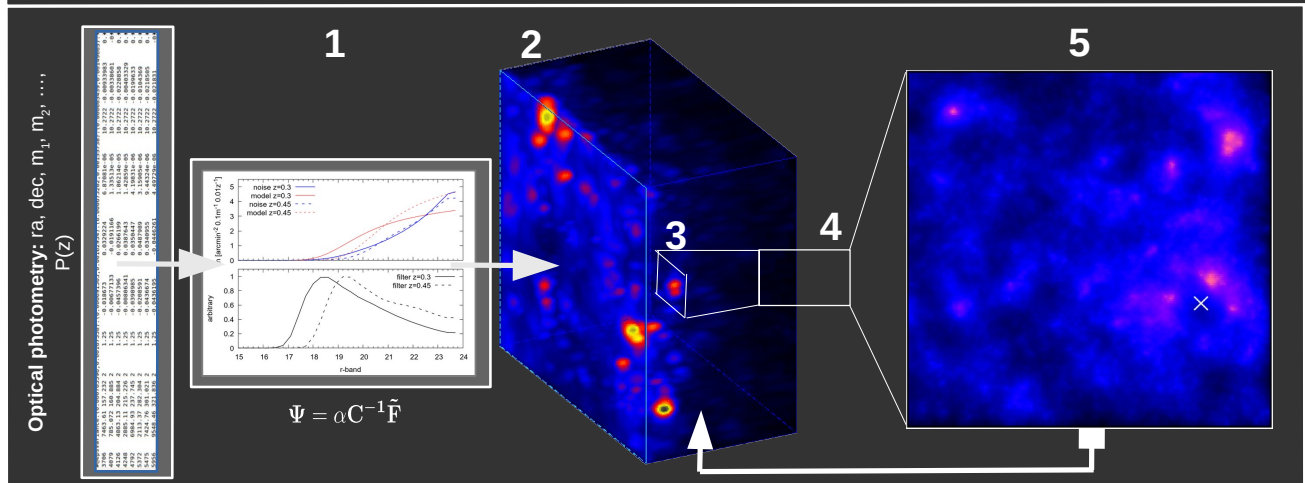
AMICO

- 1) Decide what you are searching for: cluster model and background as a function of z (*the filter*)
 - 2) Make amplitude 3D map with its uncertainty and likelihood
 - 3) Select the most-likely detection among pixels with S/N above threshold
 - 4) Attribute membership probability to galaxies
 - 5) Cleaning map from detection by removing contribution of member galaxies
- Repeat down to SN_{min}



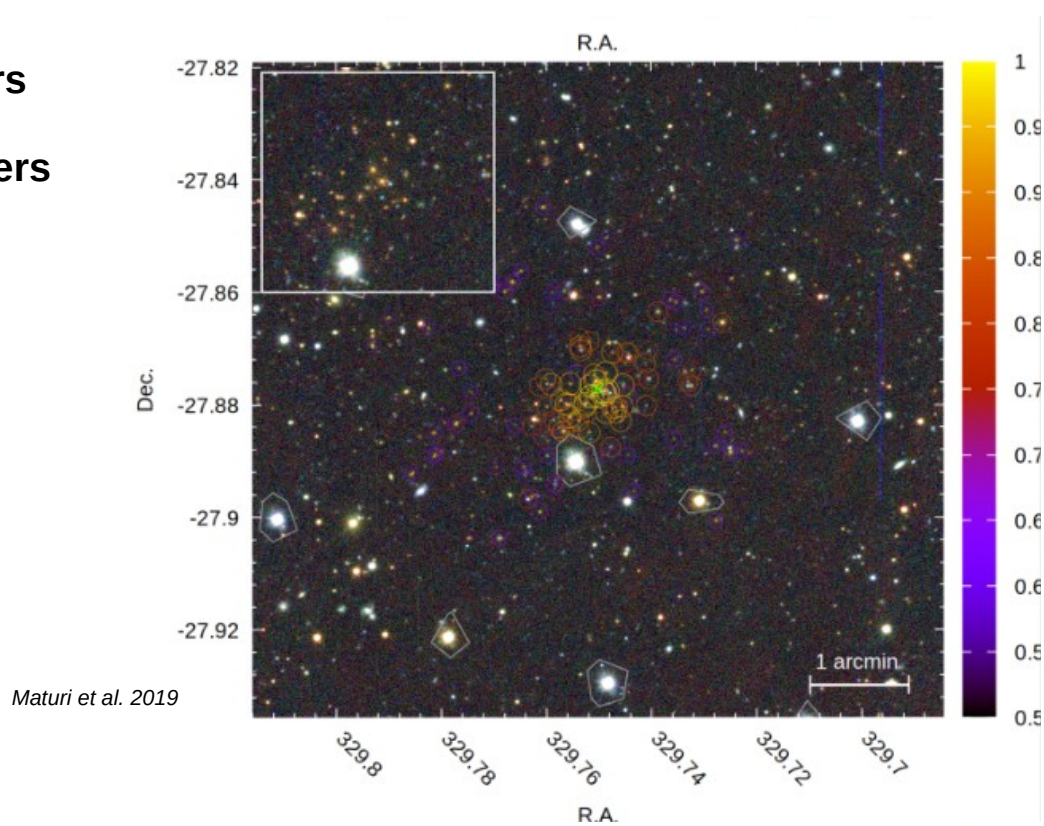
AMICO

- 1) Decide what you are searching for: cluster model and background as a function of z (*the filter*)
- 2) Make amplitude 3D map with its uncertainty and likelihood
- 3) Select the most-likely detection among pixels with S/N above threshold
- 4) Attribute membership probability to galaxies
- 5) Cleaning map from detection by removing contribution of member galaxies

Repeat down to SN_{min} 

You get the clusters

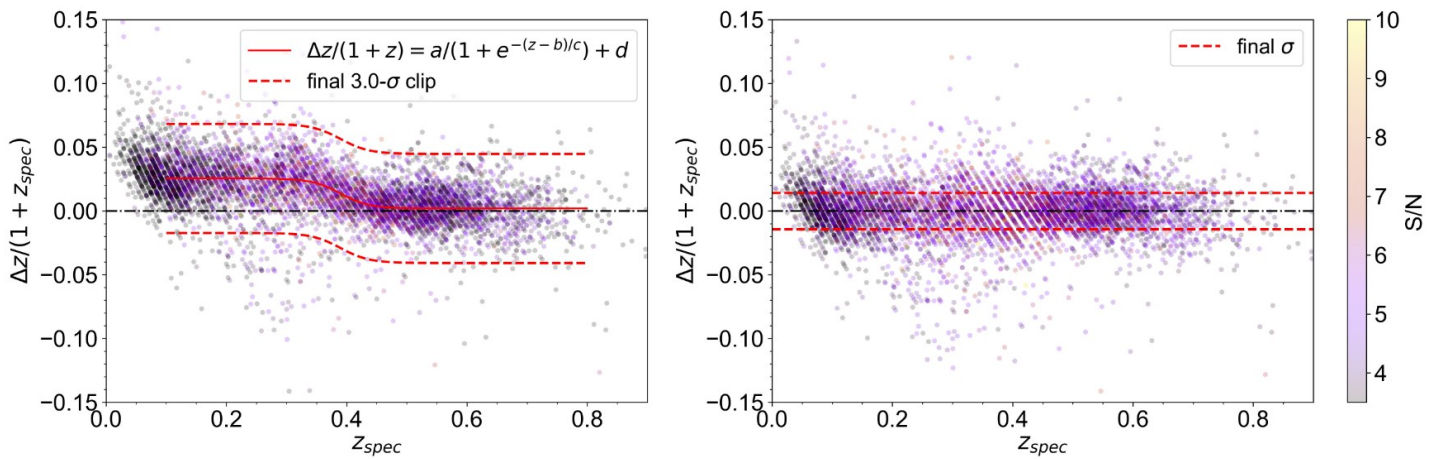
You get the members



You get a redshift estimate

e.g. AMICO-KiDS-1000 v.s. DESI + GAMA + KiDZ

Calibration with spec-z is best



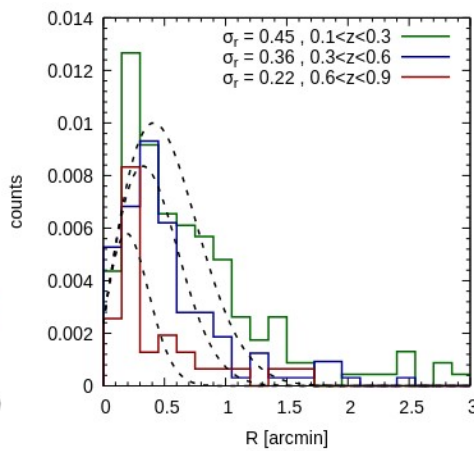
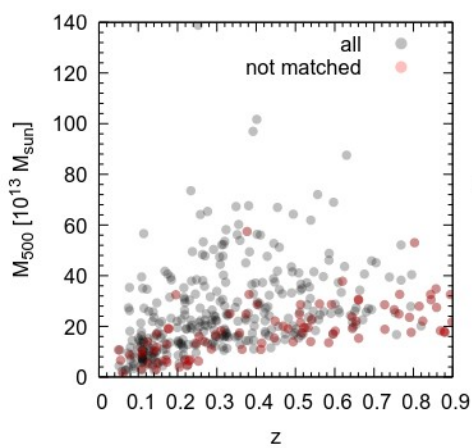
Maturi et al. 2025

You get a mass proxy: richness

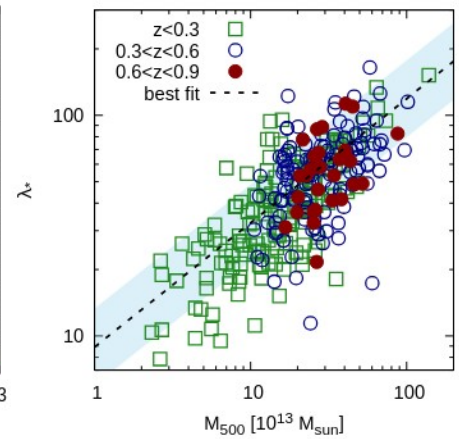
Careful, several definitions!

e.g. AMICO-KiDS-1000 v.s. eRASS1

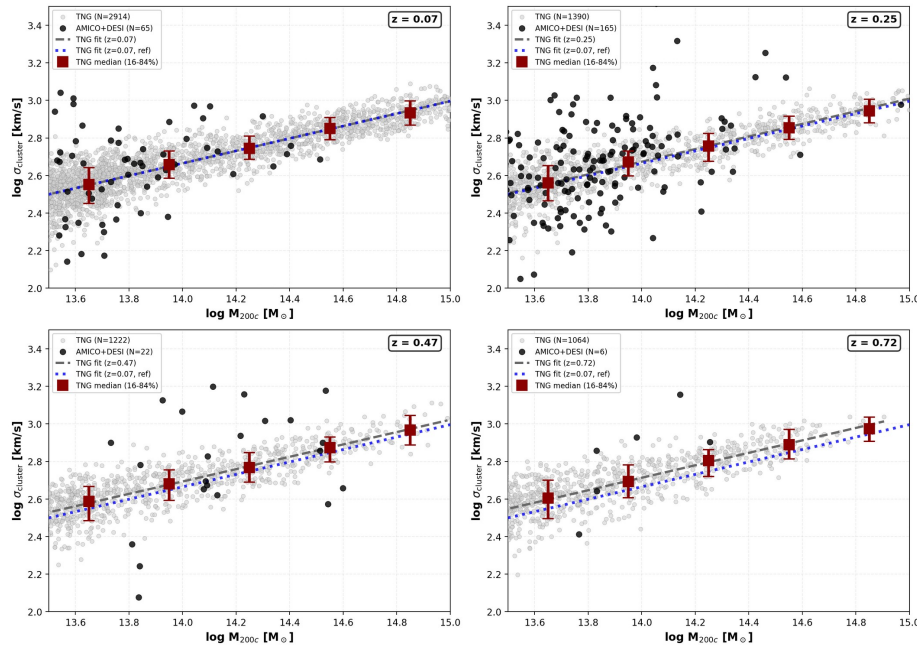
In reality Masses are from Lensing (optical) and Spectroscopy



KiDS-1000 Maturi et al. 2025



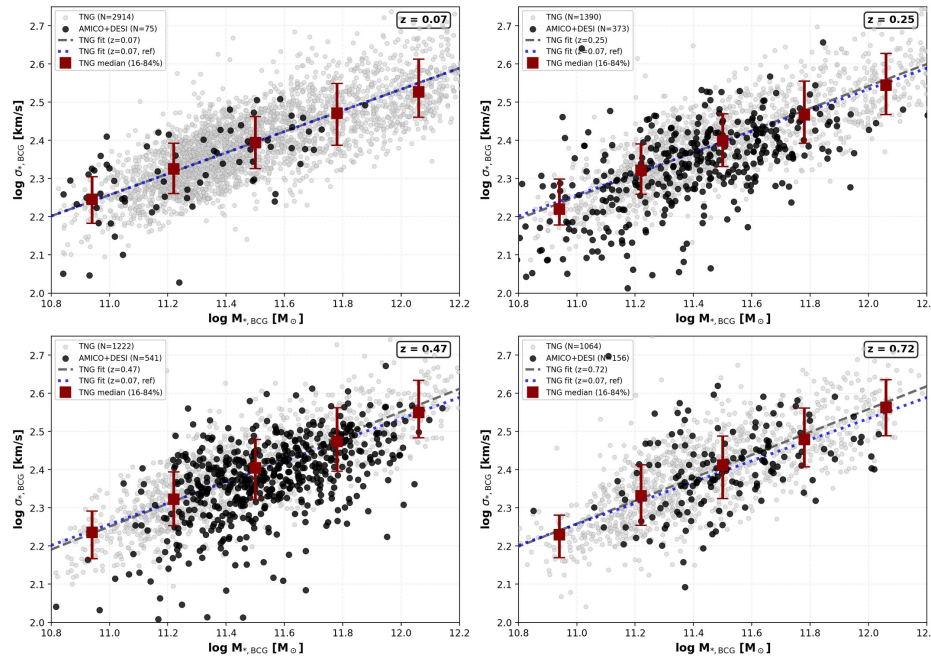
Optical/NIR Spectroscopy gives you masses as well



- DESI-DR1 GAMA-DR4 spec-z
- TNG300-1 and TNG-Cluster
- Good agreement with the values from TNG simulations

Radovich in prep.

...and link to BCGs

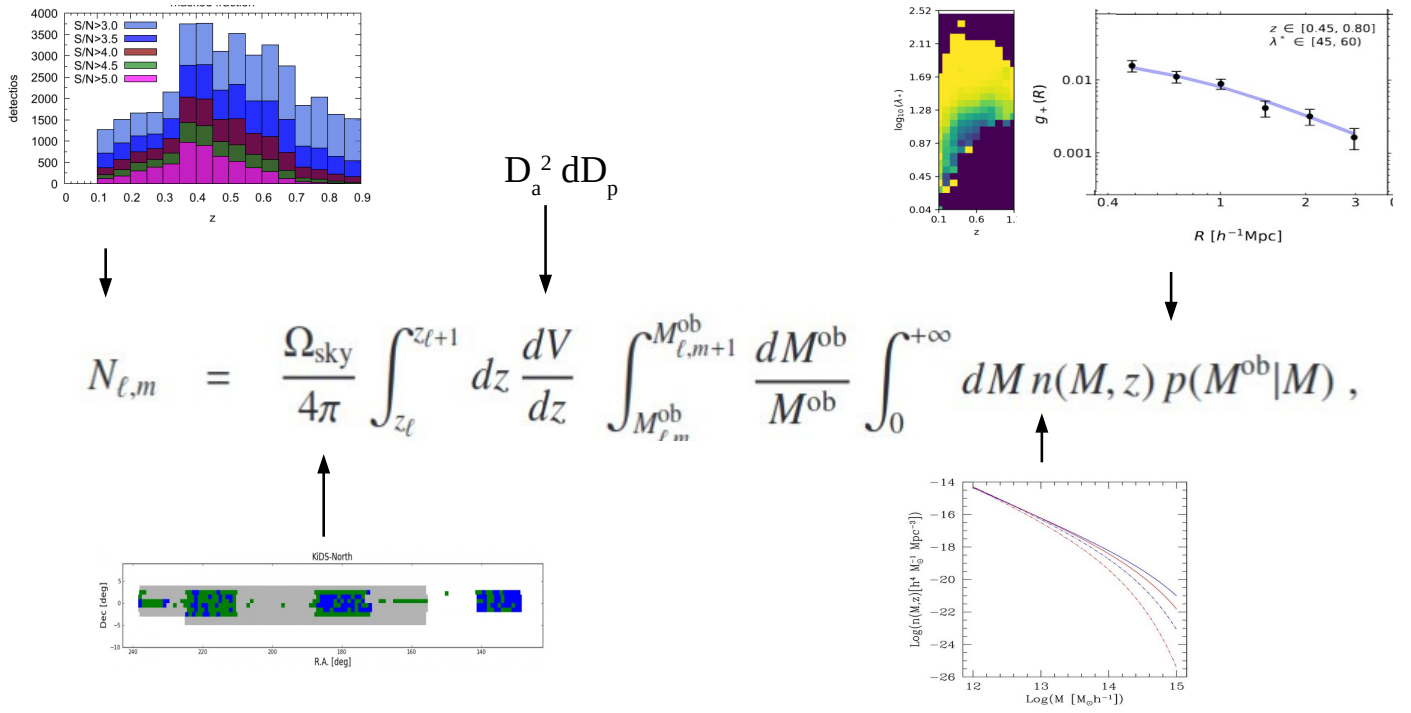


- Scatter of the BCG stellar velocity dispersion increases with z suggesting an increased dynamical complexity of clusters at higher z (see also Sohn+2022, 2024)

Radovich in prep.

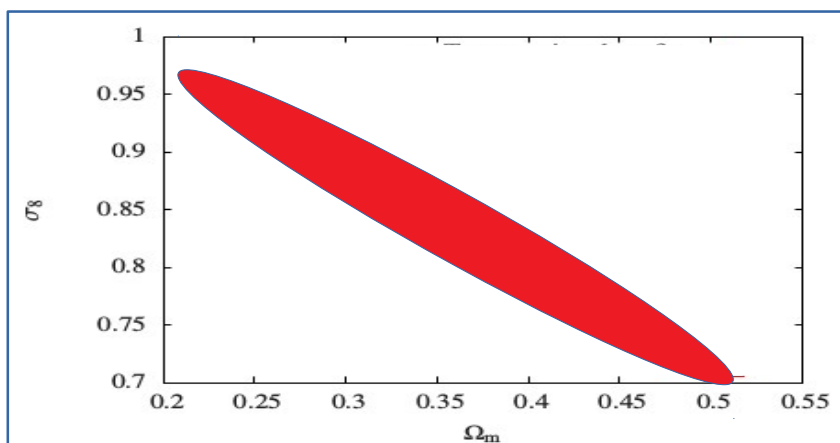
Cosmological analysis

Cosmological measure: counts



Cosmology with galaxy clusters

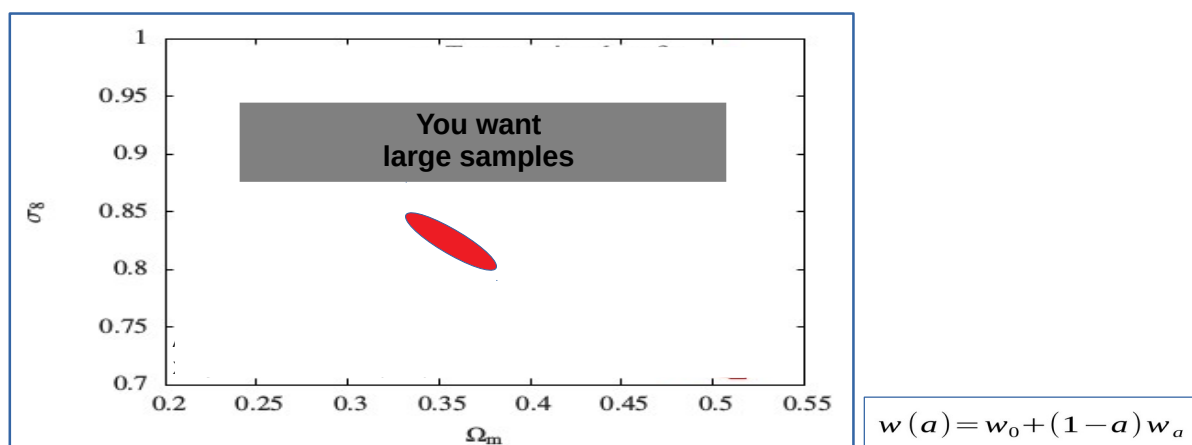
- We look for large samples: large volumes, small masses, high z
- But what you want in the end is:



$$w(a) = w_0 + (1 - a)w_a$$

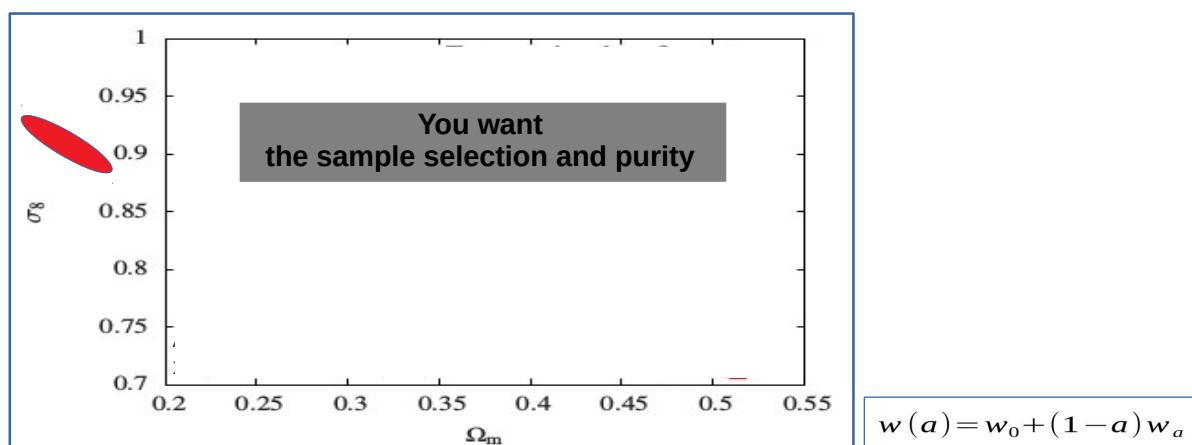
Cosmology with galaxy clusters

- We look for large samples: large volumes, small masses, high z
- But what you want in the end is:



Cosmology with galaxy clusters

- We look for large samples: large volumes, small masses, high z
- But what you want in the end is:



Sample statistical properties

Selection function: how many cluster do we miss? $P(z_{\text{true}}, \text{rich}_{\text{true}})$

Sample purity: how many detections are fake? $F(z_{\text{obs}}, \text{rich}_{\text{obs}})$

1) Validation with external data sets: X-ray, SZ, Spec-z

- sorted-2D Matching. Not trivial with high density of clusters
- Limited by sample selection of external sample

2) Numerical simulations

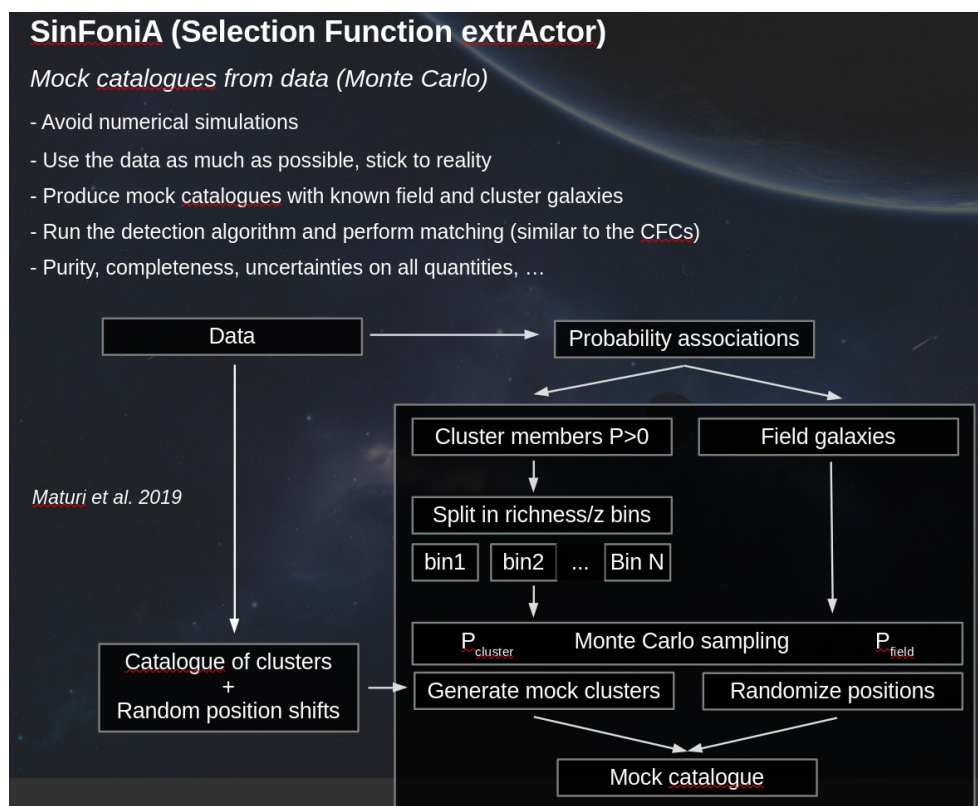
- Ground truth to compare with
- Cluster members in simulations are difficult

3) Injection in data

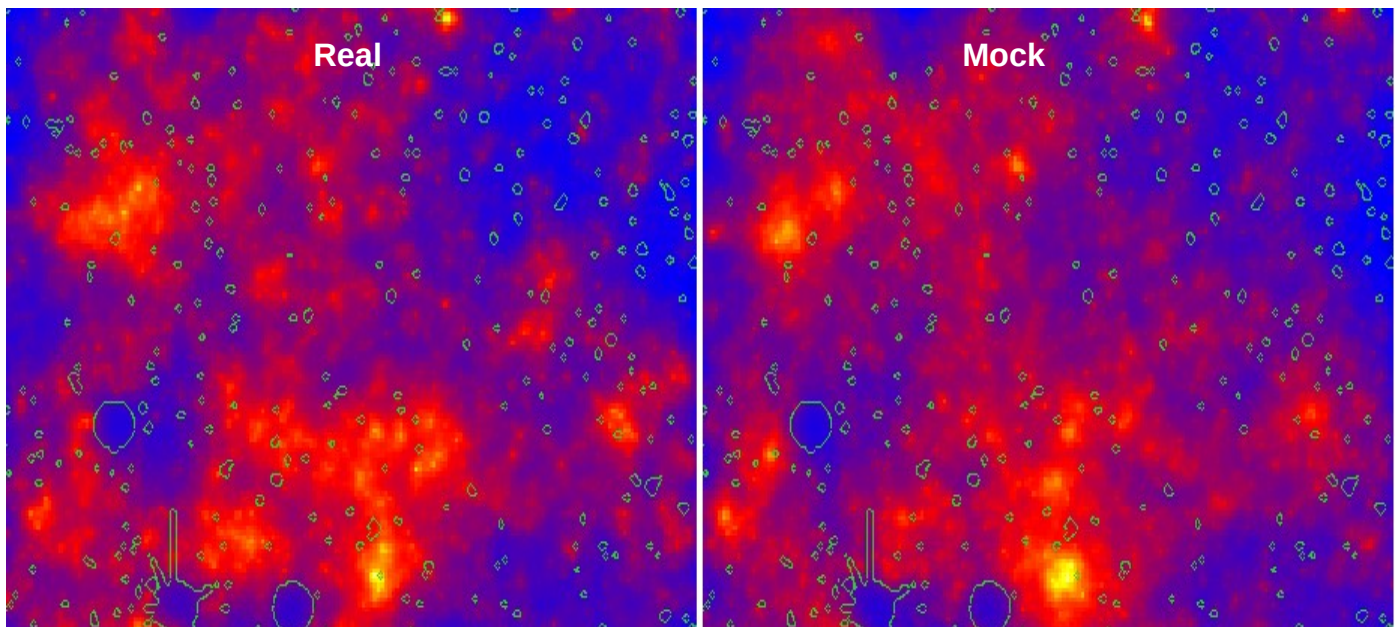
- Recipe for Member galaxies
- Plug at random in data

4) Data Driven approach

- Minimal assumptions
- exploiting the probabilistic memberships for mock construction



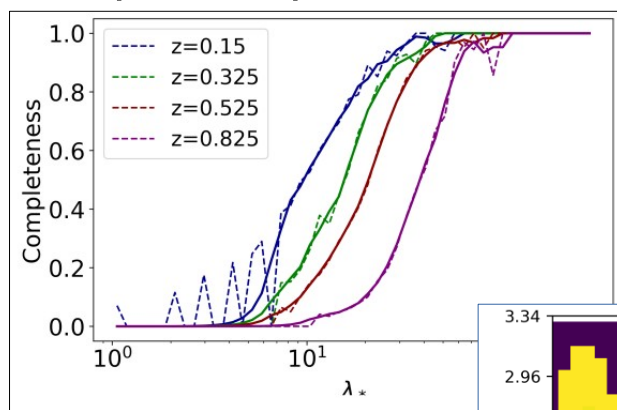
Mocks (data driven) + cluster finder



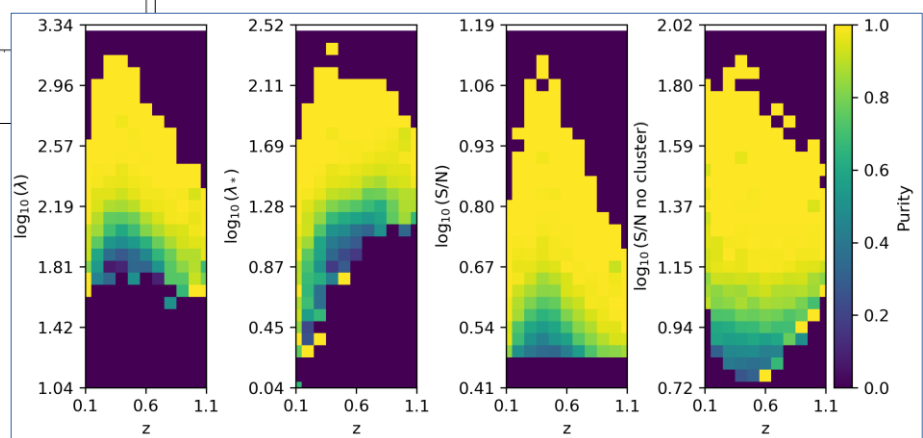
An example: one of the KiDS tiles

Maturi et al. 2019

Sample completeness



Sample purity



Maturi et al. 2025

Cosmological measure: a sketch...

$$N_{\ell,m} = \frac{\Omega_{\text{sky}}}{4\pi} \int_{z_\ell}^{z_{\ell+1}} dz \frac{dV}{dz} \int_{M_{\ell,m}^{\text{ob}}}^{M_{\ell,m+1}^{\text{ob}}} \frac{dM^{\text{ob}}}{M^{\text{ob}}} \int_0^{+\infty} dM n(M, z) p(M^{\text{ob}}|M) ,$$

↑
Mass!!!

The mass is not an observable!

What we have is richness

Weak lensing measures (optical)



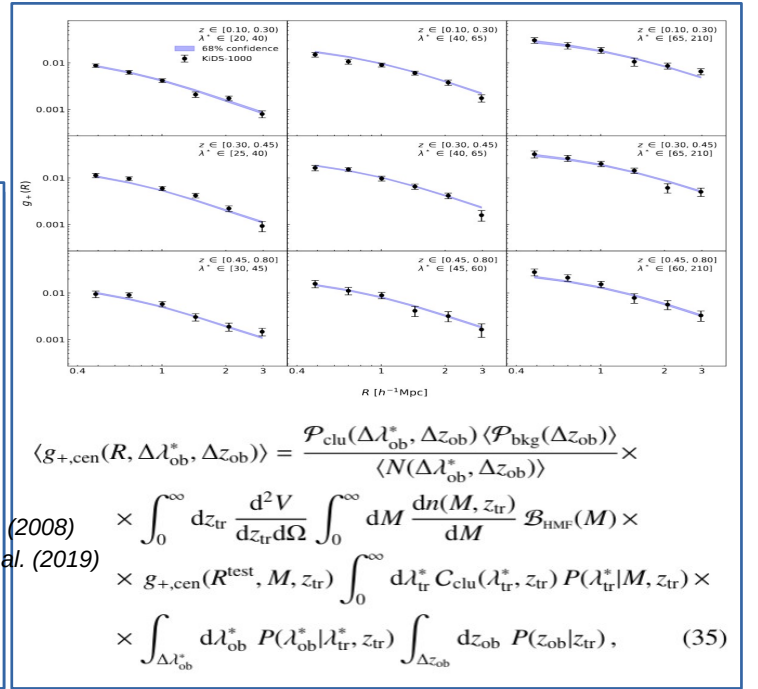
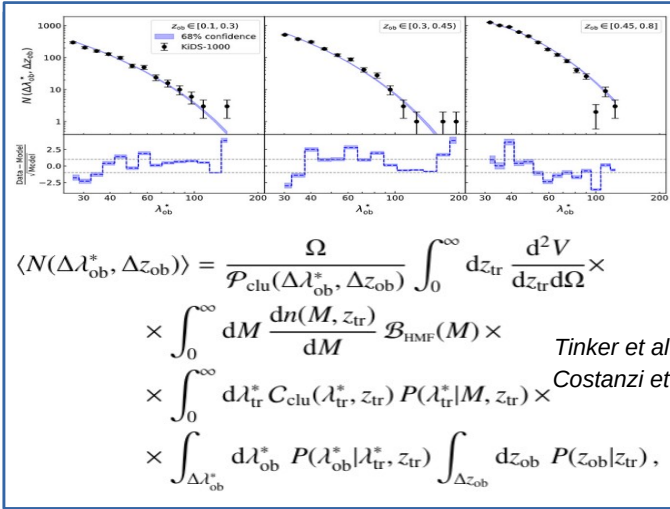
Mass-observable scaling relation: $M=m(\text{rich}, z)$

Cosmology – Scaling relation simultaneous fit

Cosmological measure: joint modeling, counts_weak lensing

$$\mathcal{L} = \mathcal{L}^{\text{WL}} \mathcal{L}^{\text{C}}$$

Lesci et al. 2025



Optical surveys are multi probes

DES Year 3 [Abbot et al. (2025)]

- $\sim 5,000 \text{ deg}^2$
- 16,000 redMaPPer clusters $z \in [0.2, 0.7]$
- $N_{\text{eff}} = 5.59 \text{ gal/arcmin}^2$ $\sigma_e = 0.261$
- Blinding on posterior
- CL+GC: Clusters counts, clustering, lens
Galaxy clustering
Cross clustering clusters-galaxy

$$S_8 = 0.864 \pm 0.035$$

$$\Omega_m = 0.265^{+0.019}_{-0.031}$$

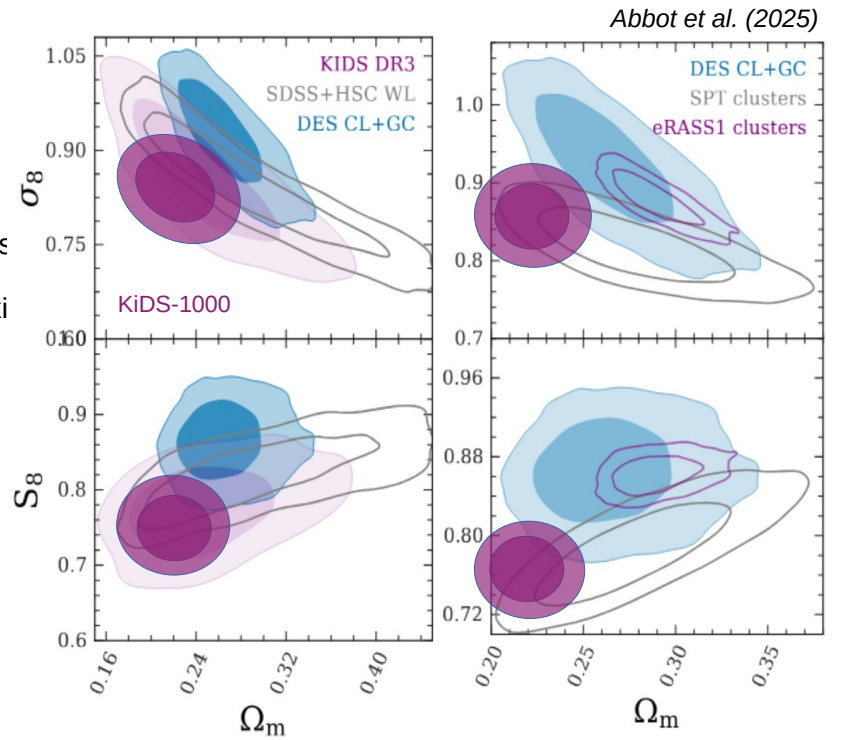
KiDS-1000 [Lesci et al. (2025)]

- 839 deg^2
- 8000 AMICO clusters $z \in [0.1, 0.8]$
- Counts + WL

$$\Omega_m = 0.218^{+0.024}_{-0.021}$$

$$S_8 = 0.86^{+0.03}_{-0.03}$$

$$S_8 = 0.74^{+0.03}_{-0.03}$$



A comparison between surveys

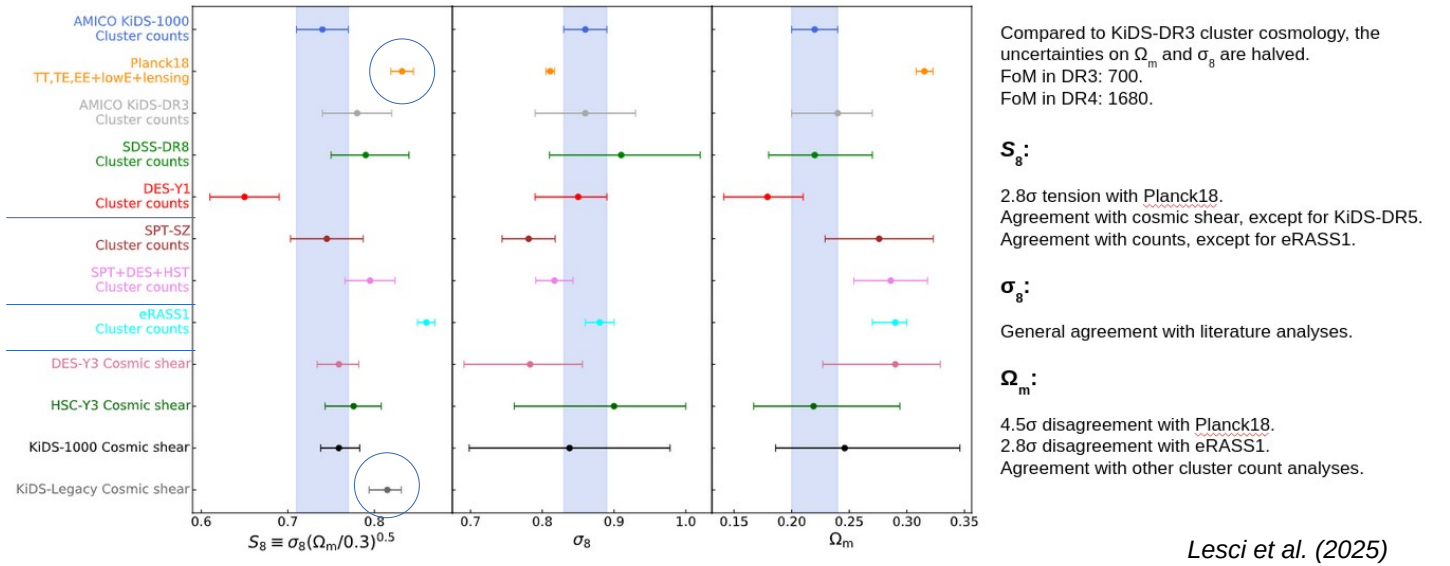
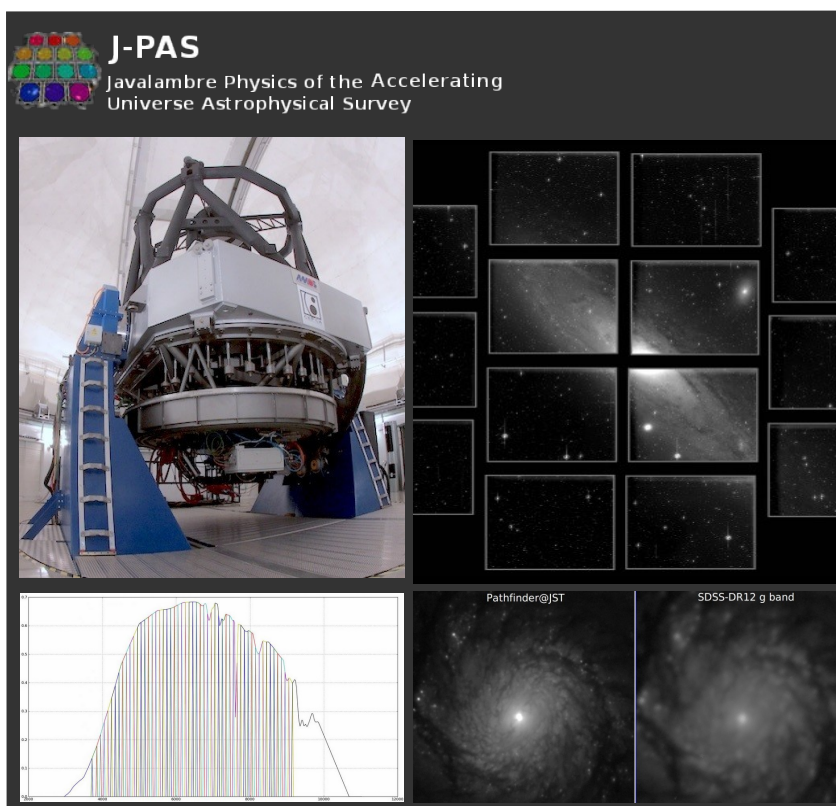


Fig. 12. Constraints on S_8 (left panel), σ_8 (middle panel), and Ω_m (right panel) obtained, from top to bottom, in this work (blue), by Planck18 (orange), Lesci et al. (2022a) (grey), Costanzi et al. (2019) (green), Abbott et al. (2020) (red), Abbott et al. (2025) (dark blue), Bocquet et al. (2019) (brown), Bocquet et al. (2024b) (violet), Ghirardini et al. (2024) (cyan), Secco et al. (2022) (pink), Dalal et al. (2023) (dark green), Asgari et al. (2021) (black), and Wright et al. (2025b) (dark grey). The median as well as the 16th and 84th percentiles are shown.

What next?

Or what now...

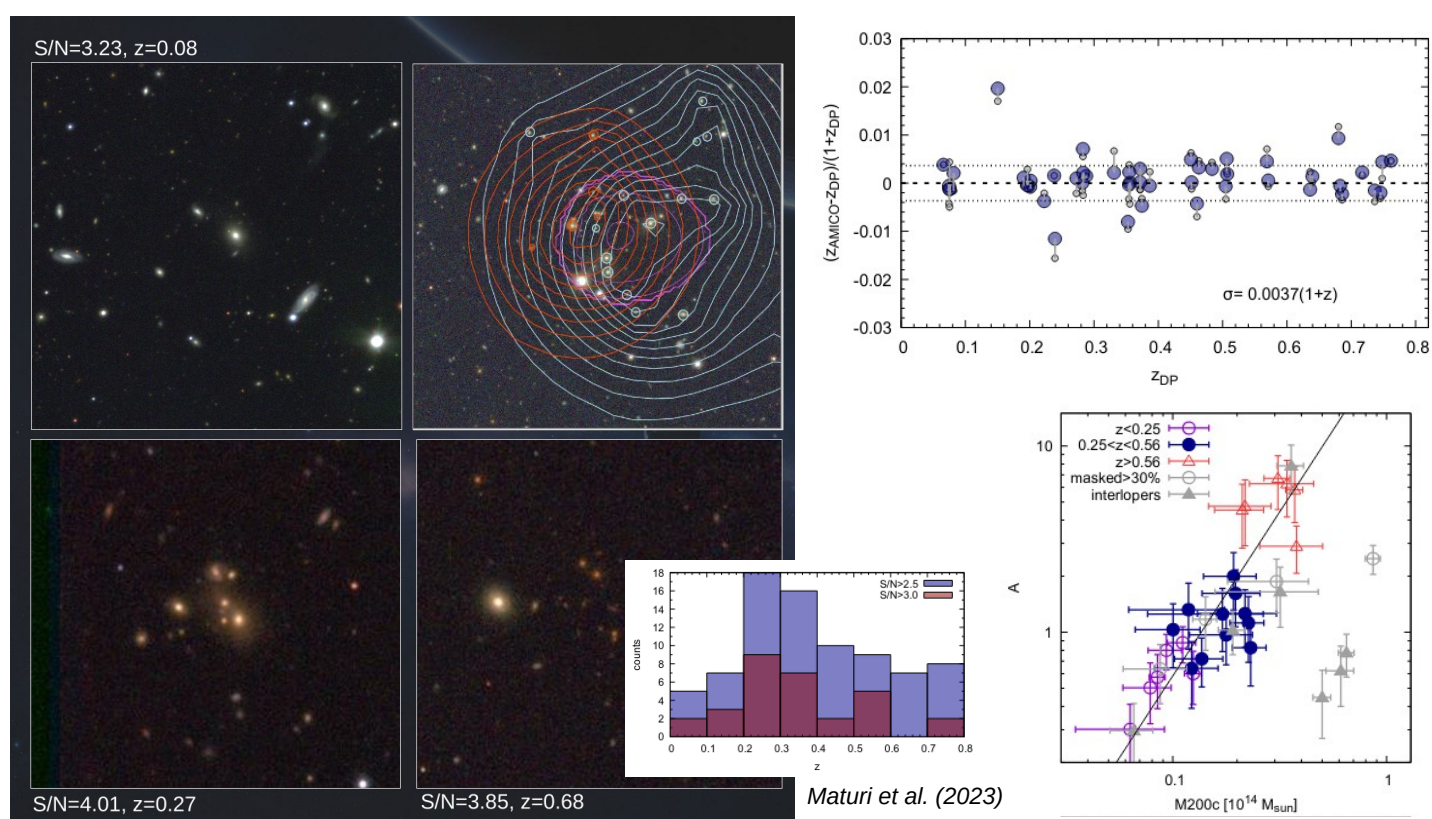


A nearly spectroscopic survey

OAJ

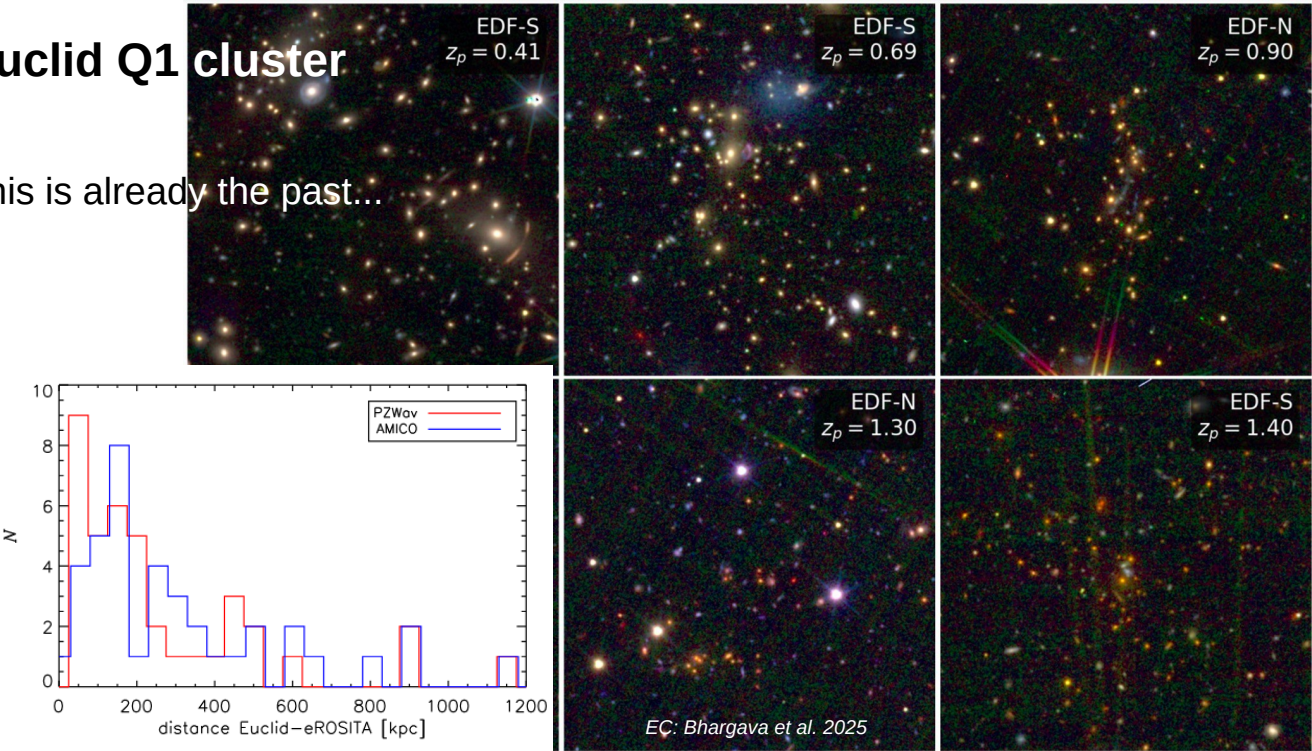
Observatorio Astrofísico de Javalambre

- Pico delBuitre of the Sierra de Javalambre, Teruel, Spain (1957m)
- JST/T250 Telescope: 2.5m
- JPCam: 7 square degrees
- 56 narrow bands + u, g, r (22.75), i
- area: 8000 deg²
- $\sigma_z \sim 0.003 (1+z)$
- Seeing (V band):
median 0.71" mode of 0.58"

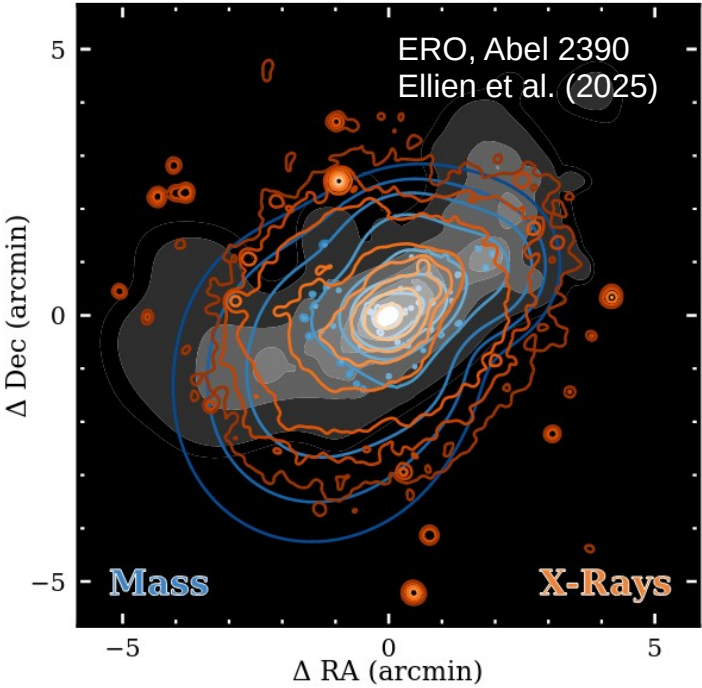
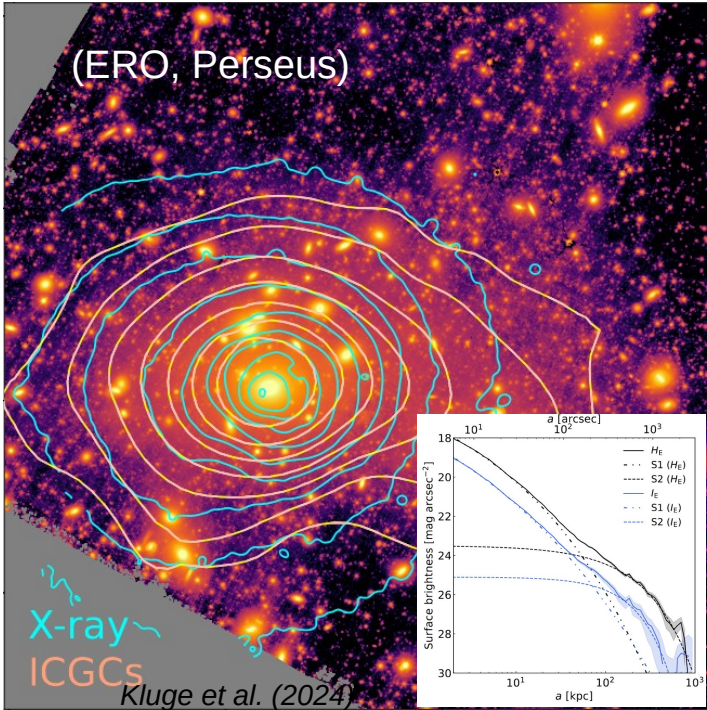


Euclid Q1 cluster

This is already the past...

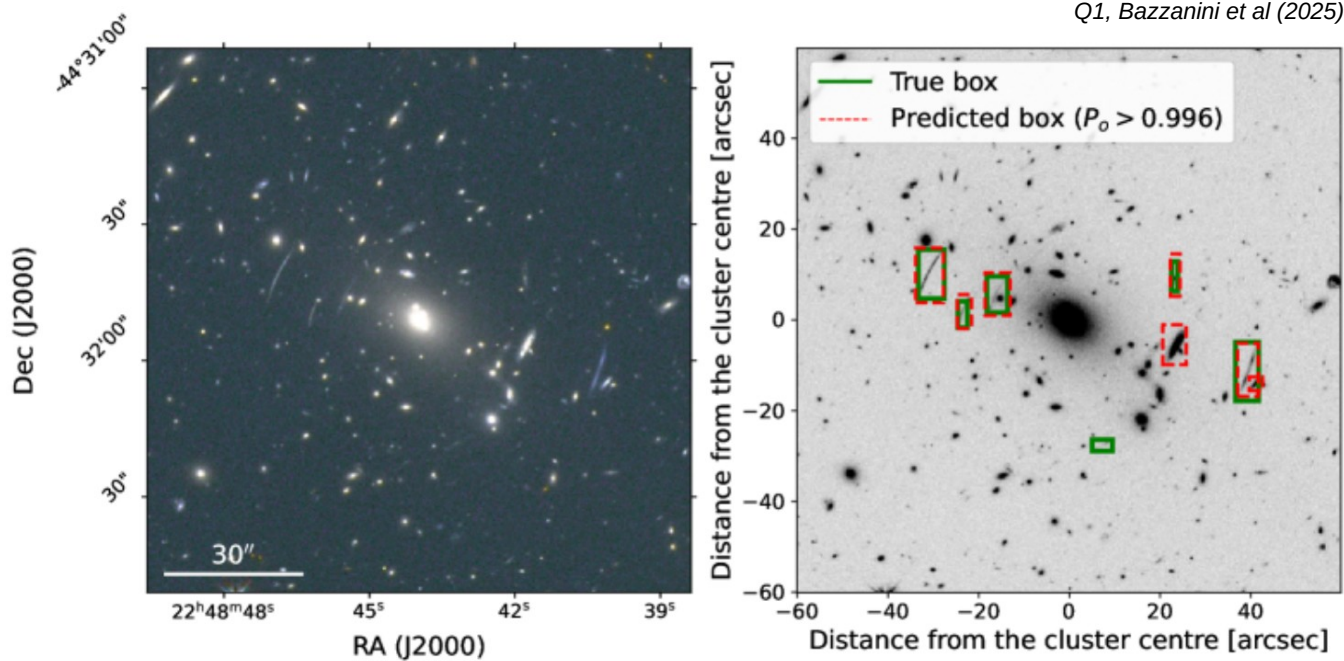


Intracluster light

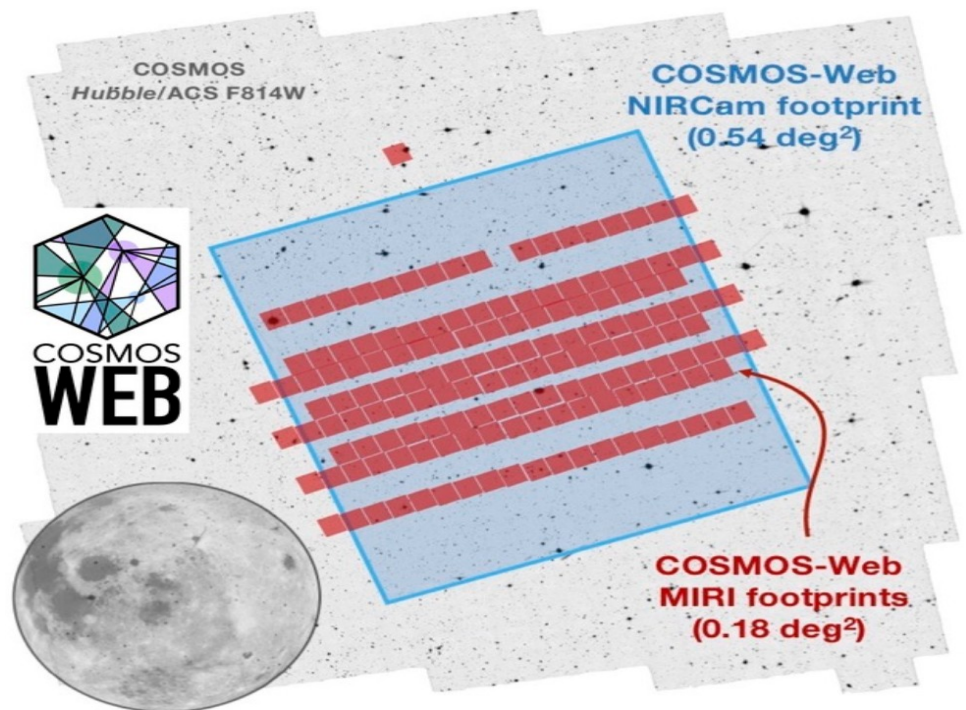


Strong lensing on an industrial scale

Q1, Bazzanini et al (2025)



COSMOS-Web

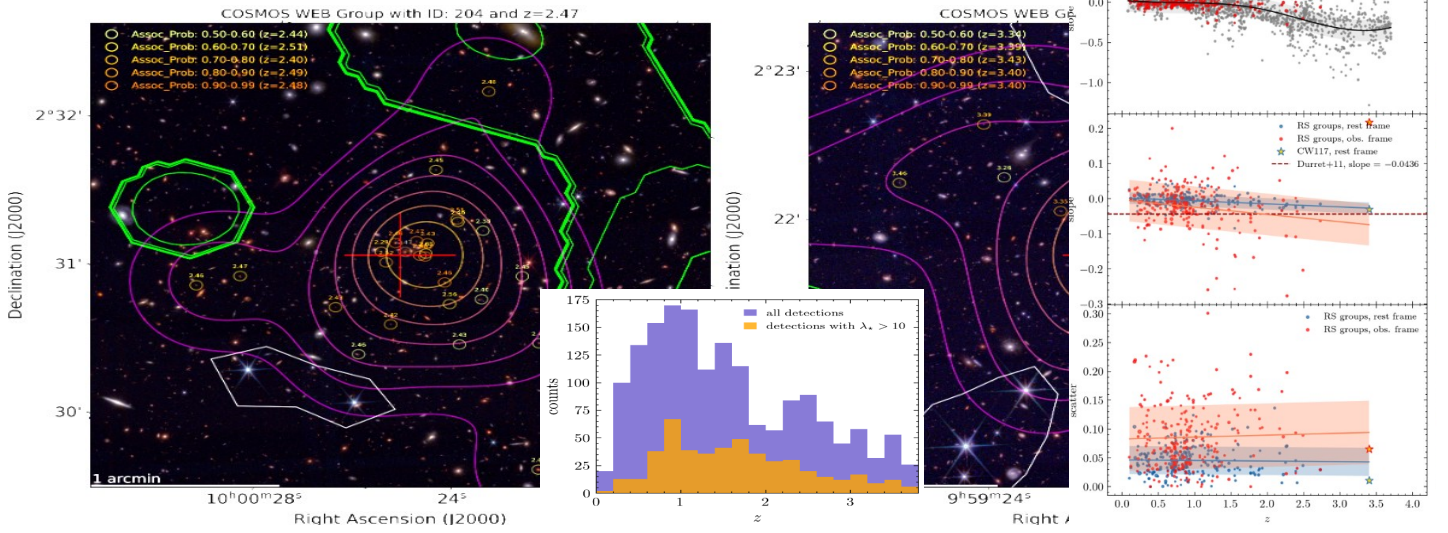


COSMOS-Web galaxy groups: Evolution of red sequence and quiescent galaxy fraction

Toni et al. et al (2025b)

The COSMOS-Web deep galaxy group catalog up to $z = 3.7$

Toni et al. et al (2025a)



Clusters are a “multi-probe” probe

Cluster counts

Clustering of galaxy clusters

Cluster structural properties

Standard ruler...

Strong lensing time delays (e.g. Refstad SN)

Indirect detection

DM self annihilation: $X + \bar{X} \rightarrow SM + \bar{SM}$ production of photons (possibly detectable) gamma

X = dark matter particle

SM = standard model particle, high energy particles: γ rays, electrons, positrons, neutrinos

γ rays would be excellent messengers

The interaction process

Reaction rate: $\Gamma = n \langle \sigma v \rangle$

$$[\Gamma] = m^3 \frac{m}{s} = s^{-1}$$

per unit volume

n = number density of DM particles

σ = interaction cross section: probability for 2 DM particles to annihilate

v = relative particles velocity

$\langle \rangle$ = average over particles velocity distribution

$\langle \sigma v \rangle$ = velocity-averaged annihilation cross section

for thermal relics (e.g. WIMPs) $\langle \sigma v \rangle \sim 10^{-26} \text{ cm}^3/\text{s}$

number of annihilations in dt : $dN = \int \Gamma dt dV = \frac{\langle \sigma v \rangle}{f} \int n_x^2(\vec{x}) dt dV = \frac{\langle \sigma v \rangle}{f m_x^2} \int g_x^2(\vec{x}) dt dV$

photon spectrum per annihilation: $\frac{dN_\gamma}{dE}$

photons production rate per unit energy and volume: $\frac{dL_\gamma}{dt dE dV} = \frac{dN}{dt dV} \frac{dN_\gamma}{dE} = \frac{\langle \sigma v \rangle}{f m_x^2} g_x^2(\vec{x}) \frac{dN_\gamma}{dE}$ (luminosity)

Observed flux per unit volume: $d\Phi_\gamma = \frac{1}{4\pi r^2} \frac{dL_\gamma}{dt dE dV} dV = \frac{1}{4\pi r^2} \frac{\langle \sigma v \rangle}{f m_x^2} g_x^2(\vec{x}) \frac{dN_\gamma}{dE} dV$

line of sight integral over an observed solid angle $\Delta\Omega$ $dV = r^2 dr d\Omega$

$$\Phi_{\Delta\Omega} = \int_{\Delta\Omega} \int_{l.o.s.}^{\infty} \frac{1}{4\pi r^2} \frac{\langle \sigma v \rangle}{f m_x^2} g_x^2(\vec{x}) \frac{dN_\gamma}{dE} r^2 dr d\Omega = \underbrace{\frac{1}{4\pi r^2} \frac{\langle \sigma v \rangle}{f m_x^2} \frac{dN_\gamma}{dE}}_{\text{physics}} \underbrace{\int_{\Delta\Omega} \int_{l.o.s.}^{\infty} g_x^2(\vec{x}) dr d\Omega}_{\text{astrophysics}}$$

$$\mathcal{J}(\Delta\Omega) = \int_{\Delta\Omega} \int_{l.o.s.}^{\infty} g_x^2(\vec{x}) dr d\Omega$$

$g(\vec{x})$ mass density spatial distribution, e.g. NFW profile

neglecting spatial dependency of $\langle \sigma v \rangle$ on position

annihilation $\mathcal{J}$ -factor models the astrophysical distribution of DM particles along a light cone

$$\text{Example } g = \text{const} \quad \mathcal{J} = \frac{4\pi R^2 g^2}{3R} \quad \text{spherical radius } R$$

f : counting factor

is needed to avoid double-counting annihilating pairs $A+B$ and $B+A$

depends on the nature of the DM particles

$f=2$ self-conjugate particle: particle = antiparticle $X+X$ (like Majorana fermions)

$f=4$ non-self-conjugate particle: particle $\neq$ antiparticle $X+\bar{X}$ (like Dirac fermions)

Statistics: basics

Random variable

x

Probability

$P(x)$

Probability distribution function (PDF)

$f(x)$

$$P(x) = f(x) dx$$

prob. to have x within dx

$$f(x) \geq 0$$

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

$$P(<B) = \int_{-\infty}^B f(x) dx$$

prob. to have a value below B

$$P(\geq B) = 1 - P(<B)$$

$$P(<B) + P(\geq B) = 1$$

$$P(\bar{A}) = 1 - P(A)$$

Probability of NOT finding A

Joint events: probability to find simultaneously (x, y)

$$P(x, y) = f(x, y) dx dy$$

$$f(x, y) \geq 0$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$$

$$p(x) = \int f(x, y) dy$$

marginalization of x over y

if x, y are independent $\Rightarrow$ marginalization is independent of y in any range

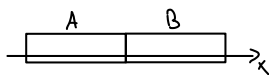
Disjoint probabilities :

A, B mutual exclusive, i.e. or A or B , not both

eg. throw a dice you get 1 number 1 or 2 for ex.

$$A, B \text{ sets} \mid A \cap B = \emptyset \quad x \in (A + B)$$

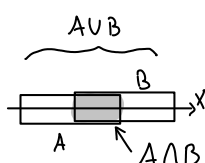
$$P(A \cup B) = P(A) + P(B)$$



$$P(x \in A + B) = \int_A f(x) dx + \int_B f(x) dx$$

$$A, B \text{ sets} \mid A \cup B \neq \emptyset \quad x \in (A + B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



$$P(x \in A + B) = \int_A f(x) dx + \int_B f(x) dx - \int_{A \cap B} f(x) dx$$

prob. to have A and B simultaneously

Joint probability

A, B independent events (i.e. 2 dices, prob. to get 2 from one and 5 from the other one)

$P = P(A)P(B)$ probability of having both A and B

$$P(A \cap B) = P(B \cap A) = P(A)P(B) \quad \cap = \text{and} \quad \cup = \text{or}$$

Conditional probability

A, B are NOT independent (the outcome of A depends on B)

$$P(A \cap B) = P(A|B) \cdot P(B)$$

probability to have simultaneously A, B

$\uparrow$ $\uparrow$
 " " " A given B conditional probability

Example:

$$P(x) = \int_{-\infty}^{\infty} f(x) dx$$

$$P(x < 0, x \in I) = \int_{-1}^0 P(x) dx \quad \text{prob. of having } x \in I = [-1, 1] \text{ and } x < 0 \text{ at the same time}$$

$$P(x \in I | x < 0) = \frac{P(x < 0, x \in I)}{P(x < 0)}$$

Bayes theorem

$$B \cap A = A \cap B \Rightarrow \begin{aligned} P(A \cap B) &= P(A|B)P(B) \\ P(B \cap A) &= P(B|A)P(A) \end{aligned} \Rightarrow \boxed{P(A|B) = \frac{P(B|A)P(A)}{P(B)}} \quad \checkmark$$

• If A, B independent, i.e. $P(A \cap B) = P(A)P(B)$

$$\Rightarrow \begin{aligned} P(A)P(B) &= P(A|B)P(B) \\ P(B)P(A) &= P(B|A)P(A) \end{aligned} \quad \boxed{\begin{aligned} P(A) &= P(A|B) \\ P(B) &= P(B|A) \end{aligned}}$$

• Marginalization: $P(B \cup \bar{B}) = 1$ Probability over all possible values
 $P(B \cup \bar{B} | A) = 1$ Probability to get any B given an A

$$\text{Bayes } P(B \cup \bar{B} | A) = \frac{P(A|B \cup \bar{B})P(B \cup \bar{B})}{P(A)} \Rightarrow \boxed{P(A) = P(A|B \cup \bar{B})}$$

Bayes theorem in terms of PDF

$$\begin{aligned} P(x) &= f_x dx \\ P(y) &= f_y dy \end{aligned} \quad \text{probability in infinitesimal intervals } dx \, dy$$

$$\text{Bayes theorem} \quad f_x(x|y) dx = \frac{f_y(y|x) dy f_x(x) dx}{f_y(y) dy} \rightarrow \boxed{\text{posterior } P(x|y) = \frac{\overset{\text{likelihood}}{L(y|x)} \overset{\text{prior}}{p(x)}}{\underset{\text{evidence}}{E(y)}}}$$

(im dx!) (im dy!)